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Spin-half fermions with mass dimension one: theory, phenomenology, and dark matter

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Abstract: We provide the first details on the unexpected theoretical discovery of a spin-one-half matter field with mass dimension one. It is based upon a complete set of dual-helicity eigenspinors of the charge conjugation operator. Due to its unusual properties with respect to charge conjugation and parity, it belongs to a non-standard Wigner class. Consequently, the theory exhibits non-locality with $(CPT)^2 = -\mathbb{I}$. We briefly discuss its relevance to the cosmological 'horizon problem'. Because the introduced fermionic field is endowed with mass dimension one, it can carry a quartic self-interaction. Its dominant interaction with known forms of matter is via Higgs, and with gravity. This aspect leads us to contemplate the new fermion as a prime dark matter candidate. Taking this suggestion seriously we study a supernova-like explosion of a galactic-mass dark matter cloud to set limits on the mass of the new particle and present a calculation on relic abundance to constrain the relevant cross-section. The analysis favours light mass (roughly 20 MeV) and relevant cross-section of about 2 pb. Similarities and differences with the WIMP and mirror matter proposals for dark matter are enumerated. In a critique of the theory we bare a hint on non-commutative aspects of spacetime, and energy-momentum space.

Keywords: dark matter, quantum field theory on curved space.

To celebrate the birthday of my wife Dr I S Ahluwalia-Khalilova
and in memory of my father Shri B S Ahluwalia (1933-1977) – dva-k.

Contents

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1. Introduction

Stars, their remnants, and gas in galaxies, contribute no more than 1% of the total cosmic matter-energy content. Several per cent more is accounted for by diffuse material pervading

intergalactic space [1]. This inventory of cosmic baryons accounts for no more than 5% of the universe. The problem was first brought to attention as early as 1933 by Zwicky [2, 3]. One now knows [4, 5] that the deficit is accounted for by non-baryonic dark matter, $\sim 25\%$, and some form of all pervading dark energy, $\sim 70\%$. That is, roughly 95% of matter–energy content of the universe is invisible and has no known, widely accepted, first-principle theoretical framework for its description. Rees [4] has described this situation as 'embarrassing'. The question we ask is: what is dark matter and why is it invisible? Here we show that a quantum field based on dual-helicity eigenspinors of spin-one-half charge conjugation operator, i.e., the operator associated with the particle–antiparticle symmetry, has precisely the property called for by the dark matter. In other words, we suggest that whatever dark matter is, one thing that seems reasonably assured is that in the low-energy limit it behaves as one of the representations of the Lorentz group. Since the known particles are described by quantum fields involving finite-dimensional representation spaces of usual Wigner classes [6–8] — with certain questions about Higgs particles being deferred to another place [9, 10] — the dark matter may belong to the yet unexplored unusual Wigner classes.

We do envisage the possibility that dark matter need not be confined to spin-one-half alone, even though the present paper focuses on this spin. Furthermore, while a vast majority of the physics community seems to be convinced of the existence of dark matter, it is important to remain open to the possibility that in part, if not in its entirety, the dark matter problem may be a reflection of the growth of the Newtonian constant at astrophysical scales [11] (see also [12–17]). Scientific caution suggests [18] that existing data be viewed with dark matter and modifications of gravity at large scales as complimentary contributors to the same data.

From a formal theoretical point of view, building on the classic works of Wigner [6, 19], this paper provides an account of our attempt to understand the particle content as implied by Poincaré spacetime symmetries. The literature on the subject has, so far, provided valuable general insights [6–8] but it lacks in specific constructs. Yet, a focus on specifics can bring about important and unexpected insights which otherwise escape [20]. It is in this latter spirit also that this paper comes into existence. A condensed version containing some of the key results is available as [21].

Another reason for which we venture to make our research notes public is the following. The assumption of locality has confined the physicists' focus to only those Wigner classes for which the charge conjugation, C, and the parity, P, operators commute for bosons, and anticommute for fermions. Yet attempts to merge the quantum, the relativistic, and the gravitational realms immediately ask for an element of non-locality which may be realized for example in the framework of field theories on non-commutative spaces (for reviews, see for example [22, 23]). Furthermore, attempts to reconcile LSND excess events [24, 25] indirectly suggest abandoning the locality requirement [26, 27]. Such a suggestion would gain strength if MiniBooNE confirms [28] the LSND result. This combined circumstance should encourage us to take a cautious walk outside the boundaries set by local relativistic-quantum field theories. Our first step in that realm constitutes preserving the Poincaré symmetries but abandoning the demand of locality. This is done by constructing and

studying a quantum field based on the above-indicated eigenspinors of the spin-$\frac{1}{2}$ charge conjugation operator C. We shall find that the quantum field so constructed is rich in structure: it belongs to the Wigner class with $[C,\ P]=0$, its propagator is not that for the Dirac field, and its mass dimension is one.

Initially, we did not set out to construct a field with the properties outlined above, or a field which would be a candidate for dark matter. Instead, we were exposed to this structure when we took an *ab initio* look at the Majorana field.

1.1 Genesis: from the Majorana field to a call for a new dark matter field

The spin-$\frac{1}{2}$ mass dimension one field came about as follows. The Majorana field is obtained by identifying $b^\dagger_h(\mathbf{p})$ with a 'phase factor $\times\ a^\dagger_h(\mathbf{p})$', in the standard Dirac field [8, 29–31]

$$\Psi(x) \;=\int \frac{d^3p}{2\pi^3}\frac{m}{p_0}\sum_{h=+,-}\Big[a_h(\mathbf{p})u_h(\mathbf{p})\mathrm{e}^{-\mathrm{i}p_\mu x^\mu}+b^\dagger_h(\mathbf{p})v_h(\mathbf{p})\mathrm{e}^{+\mathrm{i}p_\mu x^\mu}\Big]\,, \tag{1.1}$$

so that the charge-conjugated $\Psi(x)$, denoted by $\Psi^c(x)$, is *physically indistinguishable* from $\Psi(x)$ itself[3]:

$$\Psi^c(x)=\mathrm{e}^{\mathrm{i}\beta}\Psi(x)\,, \tag{1.2}$$

where $\beta\in\mathbb{R}$. This single observation has inspired a whole generation of physicists to devote their entire academic lives to confirm experimentally the realization of this suggestion [33]. After decades of pioneering work, the Heidelberg–Moscow (HM) collaboration has, in the last few years, presented first experimental evidence — or, as some may prefer to say, tantalizing hints — for a Majorana particle. The initial 3-σ signal now has better than 4-σ significance [34–37]. The field $\Psi(x)$ carries mass dimension three-halves.

Now, whether one is considering the Majorana field or the Dirac field, both are based upon the Dirac spinors. In 1957, there was an effort to reformulate the Majorana field [38, 39] in such a way that the new field was based upon what are known as Majorana spinors. It seems to have remained unasked as to what effect the choice for the helicity structure of the Majorana spinors has on the physical content of the resulting field, and why the same spinors should not be asked to satisfy an appropriate completeness relation in the $(1/2,0)\oplus(0,1/2)$ representation space. In the context of generalization to higher spins, a preliminary exploration of these issues emerged in [40, 41]. The unexpected results that we present here arose when the present authors decided to take the research notes contained in [42] to their logical conclusion. It turned out, as the reader will read below, that a field based on the dual-helicity eigenspinors of the spin-one-half charge conjugation operator — constituting a significant extension of the original Majorana idea — did not carry the property required for the identification with neutrinos.

In the meantime, there has been progress on the experimental front. While the evidence for a *Majorana particle* constitutes phenomenological realization of a quantum field which was never before known to have been used by Nature, the concurrent discovery — awaiting

[3]We follow, unless stated otherwise, the notation of Ryder [32].

due confirmation by other groups — that there exists a 6.3 σ DAMA-signal for *dark matter* [43] adds to the excitement. This also asks for a quantum field beyond the Standard Model. If the Majorana field was theoretically known since 1937, then, within the framework of known spacetime symmetries — with parity, and combined operation of charge conjugation and parity, violated — there is no first-principle quantum field which fits the 1933 Zwicky call of dark matter discovery. Only now, some seven decades later, are the experiments, the observations, and the theory merging with a call for a new (or a set of new) quantum field(s) which may attend to observations and experiments on dark matter.

In the context of extended spacetime symmetries, supersymmetric partners of the Standard Model fields also provide dark matter candidates (see, e.g., [44–46]); the most discussed being neutralino (see, e.g., [47–49]). But in doing that, one goes beyond the experimentally observed spacetime symmetries. If supersymmetry is discovered at LHC in a few years then our proposal will compete for a 'natural status' as a candidate for dark matter. Obviously, it is also conceivable that both supersymmetric partners *and* the construct presented here may be the source of dark matter. The remarks on mirror matter require a more detailed discussion. These are postponed to section 10.3, while the reader is referred to [50] for a recent review on dark matter.

1.2 The new spin-$\frac{1}{2}$ quantum field

In general, the charge and charge conjugation operators do not commute [51]. The Dirac particles are eigenstates of the charge operator. This fact, combined with the circumstances summarized above, suggests studying in detail the unexplored Wigner classes. The simplest of these is the spin-$\frac{1}{2}$ field

$$\eta(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\, m\, E(\mathbf{p})}} \sum_\beta \left[c_\beta(\mathbf{p}) \lambda^{\mathrm{S}}_\beta(\mathbf{p}) \mathrm{e}^{-\mathrm{i} p_\mu x^\mu} + d^\dagger_\beta(\mathbf{p}) \lambda^{\mathrm{A}}_\beta(\mathbf{p}) \mathrm{e}^{+\mathrm{i} p_\mu x^\mu} \right] , \tag{1.3}$$

where the $\lambda^{\mathrm{S/A}}(\mathbf{p})$ are the dual-helicity eigenspinors of the $(1/2,0) \oplus (0,1/2)$ charge conjugation operator (see section 3). We shall abbreviate $\lambda^{\mathrm{S/A}}(\mathbf{p})$ as *Elko* for the following reason. At the end of our path to obtain a meaningful and phonetically viable acronym, we eventually settled for the German *Elko*: **E**igenspinoren des **L**adungs**k**onjugations**o**perators.

As will be shown in detail below, the charged field $\eta(x)$ is different from that of Dirac. On identification of $d^\dagger_\beta(\mathbf{p})$ with $c^\dagger_\beta(\mathbf{p})$ up to a possible phase, it yields a neutral field which is different from that of Majorana. As already noted, our initial motivation was to offer $\eta(x)$ as a competing candidate for the Majorana field. However, an extended and detailed analysis revealed that the new field, whether charged or neutral, carries mass dimension one, and not three-halves. As such, it cannot be part of the $SU(2)_L$ doublets of the Standard Model which necessarily include spin-$\frac{1}{2}$ particles of mass dimension three-halves. In other words, a description of neutrinos by $\eta(x)$, with $d^\dagger_\beta(\mathbf{p})$ and $c^\dagger_\beta(\mathbf{p})$ identified with each other appropriately, results in the mixing of mass dimension $\frac{3}{2}$ and 1 spin-$\frac{1}{2}$ fermionic fields.[4] So, we concluded that $\eta(x)$ is not a good candidate for identification with the

[4] In this paper, while referring to a quantum field, we shall often take liberty of just saying 'dimension' rather than 'mass dimension'. With a minor exception in section 11, all our considerations are confined to the physical four-dimensional spacetime of special relativity.

electroweak neutrinos[5].

Given the possible phenomenological and theoretical importance of the results obtained, a natural question may arise in the mind of our reader as to why such a construct has not been undertaken before. One reason may be that any student of physics who wished to venture on such a journey would be immediately discouraged by knowledgeable physicists citing an important 1966 paper of Lee and Wick which essentially assures that any such theory will be non-local [7]. Yet, for the present authors, that has not been a discouragement. We simply set out to look at an alternative to the Dirac construct for reasons we have already mentioned. The feature of neutrality allows one to argue naively why such fields are likely to exhibit non-locality: typically, what localizes otherwise extended field configurations like solitons is a conserved (topological) charge (see for example [52]). In the absence of such charges there is nothing that protects the 'particle' from spreading and thus the emergence of non-locality it is not completely surprising[6]. Concurrently, it may be noted that today the conventional wisdom has evolved to a position where non-locality, and at times even CPT violation, is recognized as an expected part of a theory of quantum gravity [26, 27, 53, 54]. For instance, an argument can be made that at the interface of quantum and gravitational realms spacetime must be non-commutative, and that non-locality must be an integral part of any field-theoretic structure. The simplest of these early arguments can be found, for example, in [55–58]. We shall be more concrete about these remarks in the concluding section.

1.3 On the presentation of the paper

The general plan of the paper is apparent from the table of contents. Yet a few specifics may be in order. To establish our notation, and to remind the reader of the relation of the particle-antiparticle symmetry with the spacetime symmetries, we present a brief review on the emergence of the charge conjugation operator in section 2. The next section presents the dual-helicity eigenspinors of the charge conjugation operator, i.e., *Elko*. An appropriate new dual for these spinors is introduced in section 3.4, while the associated orthonormality and completeness relations are the subject of a short section, 3.5. The action of the $(1/2, 0) \oplus (0, 1/2)$ parity operator on the *Elko* is far from trivial, and we take some time to present the details in section 4.1. Apart from establishing that the charge conjugation and parity operators commute while acting on *Elko*, we show that square of the parity operator on the *Elko* basis is not an identity operator; instead it is given by *minus* the identity operator. Similarly, section 4.2 shows that while acting on *Elko* the square of the combined operation of charge conjugation, spatial parity, and time reversal

[5] Although we will not consider this possibility in the current work it is to be noted that, in principle, a right-handed neutrino may be *Elko*, because it does not have a charge with respect to any of the Standard Model gauge groups and thus is a truly neutral particle. When coupling to the left-handed sector with Yukawa-like terms it is emphasized that the coupling constant in such terms will cease to be dimensionless; rather, it will have positive mass dimension 1/2. Naive power counting suggests that quartic *Elko* terms may also be of importance. These considerations may be of relevance for the understanding of neutrino oscillations and neutrino mass generation and deserve a separate study.

[6] However, it is emphasized that in the context of solitons 'non-locality' refers to a classical field configuration, while the non-locality encountered in the present work appears at the level of field (anti) commutators.

operators yields *minus* the identity operator. Section 5 is devoted to a detailed examination of *Elko* at the representation-space level. The dimension one aspect of the quantum field based upon *Elko* is presented in detail in section 6. The discussion of section 5 and 6 also serves another useful purpose. It sheds additional light on the Dirac construct. The statistics for the *Elko* quantum field is the subject of section 7. Locality structure of the theory is obtained in section 8. Section 8.3 outlines elements of S–matrix theory for *Elko* and briefly discusses relevance of the obtained non-locality to the horizon problem of cosmology. Section 9 is devoted to a possible identification of the *Elko* framework to dark matter. Section 10 is focused on constraining the *Elko* mass and the relevant cross-section. The presented construct carries some similarities, and important differences, from the WIMP and mirror matter proposals. This is the subject of sections 10.1–10.3. The dual-helicity of *Elko* states gives rise to an important asymmetry. This is discussed in section 11. The emergent *Elko* non-locality is discussed in section 12, which also contains a detailed critique, and discussion pointing towards a non-commutative energy momentum space on the one hand and a non-commutative spacetime on the other hand. Section 12.5 provides a reference guide to some of the key equations; by following these equations a reader should be able to construct a rough and quick overview of the theoretical flow. The unconventionally long section 12 ends with a summary. A set of appendices provides auxiliary details of calculations and some additional *elkological* properties.

In order not to allow the discussion to spread over too large a technical landscape we have chosen to confine ourselves to the mass dimension 1 neutral, rather than charged, field. For a similar reason we shall confine ourselves to spin-$\frac{1}{2}$. Yet, we shall phrase our arguments and presentation in such a manner that the two-fold generalization, i.e., to higher spins and to charged fields, will be rendered obvious.

The subject matter at hand requires a somewhat pedagogic approach to the presentation. We follow this demand without apology, even at the cost of seeming pedantic. The reader is requested to reserve judgment until having read the entire paper and is, in particular, asked to refrain from prematurely invoking any folklore.

2. Emergence of the charge conjugation operator: a brief review

2.1 The Dirac construct

Both the Dirac and Majorana fields are built upon Dirac spinors. A Dirac spinor, in Weyl representation, is

$$\psi(\mathbf{p}) = \begin{pmatrix} \phi_{\mathrm{R}}(\mathbf{p}) \\ \phi_{\mathrm{L}}(\mathbf{p}) \end{pmatrix}, \tag{2.1}$$

where the massive Weyl spinors $\phi_{\mathrm{R}}(\mathbf{p})$ transform as $(1/2, 0)$ representation-space objects, and massive Weyl spinors $\phi_{\mathrm{L}}(\mathbf{p})$ transform as $(0, 1/2)$ representation-space objects. The momentum–space wave equation satisfied by $\psi(\mathbf{p})$ thus constructed follows uniquely [32, 59, 60] from the interplay of $\phi_{\mathrm{R}}(\mathbf{0}) = \pm\phi_{\mathrm{L}}(\mathbf{0})$ and $\phi_{\mathrm{R}}(\mathbf{p}) = \kappa^{(1/2,0)}\phi_{\mathrm{R}}(\mathbf{0})$ & $\phi_{\mathrm{L}}(\mathbf{p}) =$

$\kappa^{(0,1/2)}\phi_{\rm L}(\mathbf{0})$, where

$$\kappa^{(1/2,0)} = \exp\left(+\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\varphi}\right) = \sqrt{\frac{E+m}{2\,m}}\left(\mathbb{I}+\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\right)\,, \tag{2.2}$$

$$\kappa^{(0,1/2)} = \exp\left(-\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\varphi}\right) = \sqrt{\frac{E+m}{2\,m}}\left(\mathbb{I}-\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\right)\,. \tag{2.3}$$

The boost parameter, $\boldsymbol{\varphi}$, is defined as

$$\cosh(\varphi) = \frac{E}{m},\quad \sinh(\varphi) = \frac{|\mathbf{p}|}{m},\quad \widehat{\boldsymbol{\varphi}} = \frac{\mathbf{p}}{|\mathbf{p}|}\,; \tag{2.4}$$

and because of the identity $\cosh^2\theta - \sinh^2\theta = 1$ encodes in it the dispersion relation

$$E^2 = \mathbf{p}^2 + m^2\,. \tag{2.5}$$

The implied wave equation is the momentum–space Dirac equation[7]

$$\left(\gamma^\mu p_\mu \mp m\mathbb{I}\right)\psi(\mathbf{p}) = 0\,. \tag{2.6}$$

Here, $\mathbb{I}$ are $n\times n$ identity matrices, their dimensionality being apparent from the context in which they appear[8]. The γ^μ have their standard Weyl-representation form:

$$\gamma^0 = \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ \mathbb{I} & \mathbb{O}\end{pmatrix}\,,\quad \gamma^i = \begin{pmatrix} \mathbb{O} & -\sigma^i \\ \sigma^i & \mathbb{O}\end{pmatrix}\,,\quad \gamma^5 = \begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & -\mathbb{I}\end{pmatrix}\,, \tag{2.7}$$

with $\gamma^5 := -{\rm i}\gamma^0\gamma^1\gamma^2\gamma^3$. For consistency of the notation, $\mathbb{O}$ here represents a $n\times n$ null matrix (in the above equation, $n=2$). Obviously, the Dirac equation has four linearly independent solutions. Letting $p_\mu = {\rm i}\partial_\mu$ *and,* $\psi(x) := \exp\left(\mp{\rm i}p_\mu x^\mu\right)\psi(\mathbf{p})$, with upper sign for particles, and lower sign for antiparticles, one obtains the configuration space Dirac equation:

$$\left({\rm i}\gamma^\mu\partial_\mu - m\mathbb{I}\right)\psi(x) = 0\,. \tag{2.8}$$

2.2 Dirac's insight: not projecting out antiparticles

One would thus be inclined to introduce, as a new assumption of the theory, that only one of two kinds of motion occurs in practice. ...

P. A. M. Dirac, Nobel Lecture,[9] 1933.

[7] This result will be derived and also given an *ab initio* and detailed attention in section 5.

[8] So, for example, in equations (2.2) and (2.3), the $\mathbb{I}$ stand for 2×2 identity matrices, while in equation (2.6) $\mathbb{I}$ is a 4×4 identity matrix.

[9] The quote is from [61]. Furthermore, it may be noted that Dirac's initial hesitation to identify the associated particle with a new particle is well documented by Schweber in [62]. In brief: reluctant to introduce a new particle, Dirac initially identified the new particle with the proton. Heisenberg, Oppenheimer, Pauli,

Following insistence on 'only two degrees of freedom for a spin one half–particle', Dirac could have proposed a constraint which projected out two of the four degrees of freedom. The fact that he could have done so in a covariant manner would have assured that no one, or hardly any one, raised an objection. Had Dirac taken that path, a local $U(1)$ gauge theory based on such a covariant framework would have lacked physical viability. It would have missed Lamb shift [63, 64], not to say antiparticles [65, 66]. The lesson is inescapable [67, 68]: one should not impose mathematical constraints on a representation space to obtain an interpretation which satisfies certain empirically untested physical intuitions, or prevalent folklore. The physical intuition may ask for avoiding the doubling of the degrees of freedom or a folklore may demand a definite spin for particles, etc. Such constraints may have a limited validity in a classical framework. But in a quantum framework, the interactions will, in general, induce transitions between the classically allowed and the classically forbidden sectors unless prohibited, by a conservation law, or a selection rule, for some reason. Here, we shall follow Dirac's insight and not project out similar — i.e., anti-self-conjugate (see below) — degrees of freedom we shall encounter[10].

The derivation of the Dirac equation as outlined here carries a quantum-mechanical aspect in allowing for the fact that the two Weyl spaces may carry a relative phase, in the sense made explicit above, and concurrently a relativistic element via the Lorentz transformation properties of the Weyl spinors. In turn the very existence of the latter depends on the existence of two spacetime $SU(2)$s, with the following generators of transformation:

$$SU(2)_{\mathrm{A}}: \qquad \mathbf{A} = \frac{1}{2}\left(\mathbf{J} + \mathrm{i}\mathbf{K}\right), \tag{2.9}$$

$$SU(2)_{\mathrm{B}}: \qquad \mathbf{B} = \frac{1}{2}\left(\mathbf{J} - \mathrm{i}\mathbf{K}\right). \tag{2.10}$$

The $\mathbf{J}$ and $\mathbf{K}$ represent the generators of rotations and boosts, respectively, for any of the relevant finite-dimensional representation spaces which may be under consideration. For $\mathbf{B} = \mathbf{0}$, and $\mathbf{J} = \boldsymbol{\sigma}/2$, we have the $(\frac{1}{2}, 0)$ right-handed Weyl space, where $\mathbf{K}$ equals $-\mathrm{i}(\boldsymbol{\sigma}/2)$. For $\mathbf{A} = \mathbf{0}$, and $\mathbf{J} = \boldsymbol{\sigma}/2$, we have the $(0, \frac{1}{2})$ left-handed Weyl space for which $\mathbf{K}$ is $+\mathrm{i}(\boldsymbol{\sigma}/2)$.

From the womb of this structure emerges a new symmetry, i.e., that of charge conju-

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Tamm, and Weyl immediately saw that such an identification was not tenable and the new particle must carry the same mass as electron, and opposite charge. By 1931 Dirac was to write so himself: 'A hole, if there were one, would be a new kind of particle, unknown to experimental physics, having the same mass and opposite charge to an electron. We may call such a particle an anti-electron'. The name 'positron' was suggested to Anderson by Watson Davis (see [62]). In the 1933 Nobel lecture Dirac unambiguously writes: 'There is one feature of these equations which I should now like to discuss, a feature which led to the prediction of the positron'.

[10] This seemingly logical position encounters an element of opposition when one applies it to a related problem of Rarita–Schwinger field [67] . In this latter context the suggestion is to consider as unphysical the practice of projecting out the lower-spin components; and to, instead, treat ψ_μ as a single physical field which carries spin-$\frac{3}{2}$ as well as spin-$\frac{1}{2}$ components. Apart from [67], recent work of Kaloshin and Lomov confirms our interpretation [68].

gation. The operator associated with this symmetry is

$$C = \begin{pmatrix} \mathbb{O} & \mathrm{i}\,\Theta \\ -\mathrm{i}\,\Theta & \mathbb{O} \end{pmatrix} K\,. \tag{2.11}$$

Here, the operator K complex conjugates any Weyl spinor that appears on its right, and Θ is the Wigner's spin-1/2 time reversal operator. We use the representation

$$\Theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,. \tag{2.12}$$

For an arbitrary spin it is defined by the property $\Theta\mathbf{J}\Theta^{-1} = -\,\mathbf{J}^*$. Equation (2.11) is deliberately written in a slightly unfamiliar form. The chosen form is justified on the following grounds, and invites the remarks:

1. Even for $j = 1/2$ we refrain from identifying Θ with '$-\,\mathrm{i}\,\sigma_2$,' as is done implicitly in all considerations on the subject — see, for example, [69] — because such an identification does not exist for higher-spin $(j,0)\oplus(0,j)$ representation spaces. The existence of a Wigner time-reversal operator for all j allows for the introduction of $(j,0)\oplus(0,j)$ *Elko* representation spaces. In this paper, however, our attention is focused on $j = 1/2$.

2. This form readily generalizes to higher spins. Furthermore, as required by the Stückelberg–Feynman interpretation of antiparticles [70, 71][11], it makes the connection between particle–antiparticle symmetry and time reversal operator manifest.

Equation (2.11) is readily seen to yield the standard form, $C = -\gamma^2 K$. The boost operator, $\kappa^{(1/2,0)}\oplus\kappa^{(0,1/2)}$, and the $(1/2,0)\oplus(0,1/2)$-space charge conjugation operator, C, commute:

$$\left[C,\ \kappa^{(1/2,0)}\oplus\kappa^{(0,1/2)}\right] = 0\,. \tag{2.13}$$

This makes the notion of particle/antiparticle frame independent[12].

So, particles and antiparticles are offsprings of a fine interplay between the quantum realm and the realm of spacetime symmetries. This brief review makes it transparent[13].

The operation of C takes, up to a spinor-dependent global phase[14], Dirac's particle spinors into Dirac's antiparticle spinors and vice versa — see equation (4.12) below. Keeping with our pedagogic style, we note: the Dirac spinors are thus not eigenspinors of the charge conjugation operator.

[11] It may be worth noting that the Stückelberg–Feynman interpretation of antiparticles ceases to be equivalent to the standard interpretation in cosmological context [72].

[12] However, in general, boosts do not leave the time-order of events unchanged. This leads to interesting paradoxes, and again this necessitates existence of antiparticles. This has been discussed elegantly in section 13 of chapter 2 of Weinberg's classic on gravitation and cosmology, and since we cannot do a better job than that the reader is referred to [73].

[13] A more formal treatment of this result can be found in the classic work of Streater and Wightman [74].

[14] The spinor dependence may be removed by appropriate redefinitions without changing the physical content of the theory.

3. Dual-helicity eigenspinors of charge conjugation operator, or Eigenspinoren des Ladungskonjugationsoperators (Elko)

We have just summarized the origin and form of the charge conjugation operator. We now proceed to obtain its eigenspinors. Towards this task one may take a direct and purely mathematical approach, or adopt a slightly indirect but physically insightful path. We shall follow the latter, and will shortly argue that if $\phi_{\mathrm{L}}(\mathbf{p})$ transforms as a left-handed spinor, then $(\zeta_\lambda\Theta)\ \phi^*_{\mathrm{L}}(\mathbf{p})$ transforms as a right-handed spinor — where ζ_λ is an unspecified phase — with a similar assertion holding true for $\phi_{\mathrm{R}}(\mathbf{p})$. This allows us to define $(1/2,0)\oplus(0,1/2)$ spinors which are different from that of Dirac — which, of course, also belong to the $(1/2,0)\oplus(0,1/2)$ representation space — and which become eigenspinors of the C operator if ζ_λ is given some specific values.

3.1 Formal structure of Elko

The details are as follows: because the boost operators written in equations (2.2), (2.3) are Hermitean and inverse to each other, we have

$$\left(\kappa^{(0,1/2)}\right)^{-1} = \left(\kappa^{(1/2,0)}\right)^{\dagger}\,, \qquad \left(\kappa^{(1/2,0)}\right)^{-1} = \left(\kappa^{(0,1/2)}\right)^{\dagger}\,. \tag{3.1}$$

Further, Θ, the Wigner's spin-1/2 time reversal operator, has the property

$$\Theta\,[\boldsymbol{\sigma}/2]\,\Theta^{-1} = -\,[\boldsymbol{\sigma}/2]^*\,. \tag{3.2}$$

When combined, these observations imply that: (a) if $\phi_{\mathrm{L}}(\mathbf{p})$ transforms as a left-handed spinor, then $(\zeta_\lambda\Theta)\ \phi^*_{\mathrm{L}}(\mathbf{p})$ transforms as a right-handed spinor — where ζ_λ is an unspecified phase; (b) if $\phi_{\mathrm{R}}(\mathbf{p})$ transforms as a right-handed spinor, then $(\zeta_\rho\Theta)^*\ \phi^*_{\mathrm{R}}(\mathbf{p})$ transforms as a left-handed spinor — where ζ_ρ is an unspecified phase. These results are in agreement with Ramond's observation in [69]. As a consequence, the following spinors belong to the $(1/2,0)\oplus(0,1/2)$ representation space:

$$\lambda(\mathbf{p}) = \begin{pmatrix} (\zeta_\lambda\Theta)\ \phi^*_{\mathrm{L}}(\mathbf{p}) \\ \phi_{\mathrm{L}}(\mathbf{p}) \end{pmatrix}, \qquad \rho(\mathbf{p}) = \begin{pmatrix} \phi_{\mathrm{R}}(\mathbf{p}) \\ (\zeta_\rho\Theta)^*\ \phi^*_{\mathrm{R}}(\mathbf{p}) \end{pmatrix}. \tag{3.3}$$

Confining ourselves to real eigenvalues (the demand of observability), these become eigenspinors of the charge conjugation operator with eigenvalues, ± 1, if the phases, ζ_λ and ζ_ρ, are restricted to the values

$$\zeta_\lambda = \pm\,\mathrm{i}\,, \qquad \zeta_\rho = \pm\,\mathrm{i}\,. \tag{3.4}$$

With this restriction imposed, we have

$$C\lambda(\mathbf{p}) = \pm\lambda(\mathbf{p})\,, \qquad C\rho(\mathbf{p}) = \pm\rho(\mathbf{p})\,. \tag{3.5}$$

The plus sign yields *self-conjugate spinors:* $\lambda^{\mathrm{S}}(\mathbf{p})$ and $\rho^{\mathrm{S}}(\mathbf{p})$. The minus sign results in the *anti-self-conjugate spinors:* $\lambda^{\mathrm{A}}(\mathbf{p})$ and $\rho^{\mathrm{A}}(\mathbf{p})$. To obtain explicit expressions for $\lambda(\mathbf{p})$ we first write the rest spinors. These are

$$\lambda^{\mathrm{S}}(\mathbf{0}) = \begin{pmatrix} +\,\mathrm{i}\,\Theta\,\phi^*_{\mathrm{L}}(\mathbf{0}) \\ \phi_{\mathrm{L}}(\mathbf{0}) \end{pmatrix}, \qquad \lambda^{\mathrm{A}}(\mathbf{0}) = \begin{pmatrix} -\,\mathrm{i}\,\Theta\,\phi^*_{\mathrm{L}}(\mathbf{0}) \\ \phi_{\mathrm{L}}(\mathbf{0}) \end{pmatrix}. \tag{3.6}$$

Next, we choose the $\phi_{\mathrm{L}}(\mathbf{0})$ to be helicity eigenstates

$$\boldsymbol{\sigma}\cdot\widehat{\mathbf{p}}\,\phi^{\pm}_{\mathrm{L}}(\mathbf{0}) = \pm\,\phi^{\pm}_{\mathrm{L}}(\mathbf{0})\,, \tag{3.7}$$

and concurrently note that

$$\boldsymbol{\sigma}\cdot\widehat{\mathbf{p}}\,\Theta\left[\phi^{\pm}_{\mathrm{L}}(\mathbf{0})\right]^{*} = \mp\,\Theta\left[\phi^{\pm}_{\mathrm{L}}(\mathbf{0})\right]^{*}\,. \tag{3.8}$$

The derivation of equation (3.8) is given in appendix A.2, while the explicit forms of $\phi^{\pm}_{\mathrm{L}}(\mathbf{0})$ are given in appendix A.1. The physical content of the result (3.8) is the following: $\Theta\left[\phi^{\pm}_{\mathrm{L}}(\mathbf{0})\right]^{*}$ has opposite helicity of $\phi^{\pm}_{\mathrm{L}}(\mathbf{0})$. Since $\boldsymbol{\sigma}\cdot\widehat{\mathbf{p}}$ commutes with the boost operator $\kappa^{(1/2,0)}$ this result holds for all $\mathbf{p}$.

3.2 Distinction between Elko and Majorana spinors

So as not to obscure the physics by notational differences, it is helpful to note — a choice we confine to this subsection only — that since $\mathrm{i}\Theta = \sigma_2$ we may write

$$\lambda(\mathbf{p}) = \begin{pmatrix} \pm\sigma_2\,\phi^{*}_{\mathrm{L}}(\mathbf{p}) \\ \phi_{\mathrm{L}}(\mathbf{p}) \end{pmatrix}\,, \qquad \rho(\mathbf{p}) = \begin{pmatrix} \phi_{\mathrm{R}}(\mathbf{p}) \\ \mp\sigma_2\,\phi^{*}_{\mathrm{R}}(\mathbf{p}) \end{pmatrix}\,,$$

where the upper sign is for self-conjugate spinors, and the lower sign yields the anti-self-conjugate spinors. We now have a choice in selecting the helicity of the $(1/2,0)$ and $(0,1/2)$ components of $\lambda(\mathbf{p})$. We find that this choice has important physical consequences for reasons which parallel Weinberg's detailed analysis of Dirac spinors (see section 5.5 of [8]). In particular, as we shall confirm, that the choice affects the parity and locality properties of the constructed field. For the moment it suffices to note that if one chooses the helicity for the $(1/2,0)$ and $(0,1/2)$ components to be *same*, then the $\lambda(\mathbf{p})$ are characterized by a single-helicity and become identical to the standard Majorana spinors (see, e.g., [30, 75]). This choice violates the spirit of the result contained in equation (3.8). We fully respect the spirit and the content of the result contained in equation (3.8) and therein lies our point of departure from Majorana spinors. That is, for our *Elko* we start with the $(0,1/2)$ component $\phi_{\mathrm{L}}(\mathbf{p})$ in one or the other helicity. Then, when constructing the $(1/2,0)$ component, $\pm\,\mathrm{i}\,\Theta\,\phi^{*}_{\mathrm{L}}(\mathbf{p})$ (or, equivalently $\pm\sigma_2\,\phi^{*}_{\mathrm{L}}(\mathbf{p})$), we take the same original $\phi_{\mathrm{L}}(\mathbf{p})$ in the same helicity, i.e., we do not flip its helicity by hand. This causes the $(1/2,0)$ transforming component to carry the opposite helicity to that of the original $\phi_{\mathrm{L}}(\mathbf{p})$. This is dictated by equation (3.8). For this reason *Elko* we consider are dual-helicity objects.

Similar remarks apply to $\rho(\mathbf{p})$, which incidentally do not constitute an independent set of *Elko*[15].

3.3 Explicit form of Elko

Having thus seen the formal structure of *Elko* it is now useful to familiarize oneself by constructing them in their fully explicit form.

[15] Section 3.2 was added to the manuscript as an answer to remarks by E. C. G. Sudarshan [76].

The results of the above discussion lead to four rest spinors. Two of which are self-conjugate,

$$\lambda^{\mathrm{S}}_{\{-,+\}}(\mathbf{0}) = \begin{pmatrix} +\,\mathrm{i}\,\Theta\,\left[\phi^{+}_{\mathrm{L}}(\mathbf{0})\right]^{*} \\ \phi^{+}_{\mathrm{L}}(\mathbf{0}) \end{pmatrix}, \qquad \lambda^{\mathrm{S}}_{\{+,-\}}(\mathbf{0}) = \begin{pmatrix} +\,\mathrm{i}\,\Theta\,\left[\phi^{-}_{\mathrm{L}}(\mathbf{0})\right]^{*} \\ \phi^{-}_{\mathrm{L}}(\mathbf{0}) \end{pmatrix}, \tag{3.9}$$

and the other two are anti-self-conjugate,

$$\lambda^{\mathrm{A}}_{\{-,+\}}(\mathbf{0}) = \begin{pmatrix} -\,\mathrm{i}\,\Theta\,\left[\phi^{+}_{\mathrm{L}}(\mathbf{0})\right]^{*} \\ \phi^{+}_{\mathrm{L}}(\mathbf{0}) \end{pmatrix}, \qquad \lambda^{\mathrm{A}}_{\{+,-\}}(\mathbf{0}) = \begin{pmatrix} -\,\mathrm{i}\,\Theta\,\left[\phi^{-}_{\mathrm{L}}(\mathbf{0})\right]^{*} \\ \phi^{-}_{\mathrm{L}}(\mathbf{0}) \end{pmatrix}. \tag{3.10}$$

The first helicity entry refers to the $(1/2,0)$ transforming component of the $\lambda(\mathbf{p})$, while the second entry encodes the helicity of the $(0,1/2)$ component. The boosted spinors are now obtained via the operation

$$\lambda_{\{h,-h\}}(\mathbf{p}) = \begin{pmatrix} \kappa^{(1/2,0)} & \mathbb{O} \\ \mathbb{O} & \kappa^{(0,1/2)} \end{pmatrix} \lambda_{\{h,-h\}}(\mathbf{0})\,. \tag{3.11}$$

In the boosts, we replace $\boldsymbol{\sigma}\cdot\mathbf{p}$ by $p\,\boldsymbol{\sigma}\cdot\widehat{\mathbf{p}}$, and then exploit equation (3.8). After simplification, equation (3.11) yields

$$\lambda^{\mathrm{S}}_{\{-,+\}}(\mathbf{p}) = \sqrt{\frac{E+m}{2\,m}}\left(1-\frac{p}{E+m}\right)\lambda^{\mathrm{S}}_{\{-,+\}}(\mathbf{0})\,, \tag{3.12}$$

which, in the massless limit, *identically vanishes,* while in the same limit

$$\lambda^{\mathrm{S}}_{\{+,-\}}(\mathbf{p}) = \sqrt{\frac{E+m}{2\,m}}\left(1+\frac{p}{E+m}\right)\lambda^{\mathrm{S}}_{\{+,-\}}(\mathbf{0}) \tag{3.13}$$

does not. We hasten to warn the reader that one should not be tempted to read the two different pre-factors to $\lambda^{\mathrm{S}}(\mathbf{0})$ in the above expressions as the boost operator that appears in equation (3.11). For one thing, there is only one (not two) boost operator(s) in the $(1/2,0)\oplus(0,1/2)$ representation space. The simplification that appears here is due to a fine interplay between equation (3.8), the boost operator, and the structure of the $\lambda^{\mathrm{S}}(\mathbf{0})$. Similarly, the anti-self-conjugate set of the boosted spinors reads

$$\lambda^{\mathrm{A}}_{\{-,+\}}(\mathbf{p}) = \sqrt{\frac{E+m}{2\,m}}\left(1-\frac{p}{E+m}\right)\lambda^{\mathrm{A}}_{\{-,+\}}(\mathbf{0})\,, \tag{3.14}$$

$$\lambda^{\mathrm{A}}_{\{+,-\}}(\mathbf{p}) = \sqrt{\frac{E+m}{2\,m}}\left(1+\frac{p}{E+m}\right)\lambda^{\mathrm{A}}_{\{+,-\}}(\mathbf{0})\,. \tag{3.15}$$

In the massless limit, the first of these spinors *identically vanishes*, while the second does not.

3.4 A new dual for Elko

For any $(1/2,0)\oplus(0,1/2)$ spinor $\xi(\mathbf{p})$, the Dirac dual spinor $\overline{\xi}(\mathbf{p})$ is defined as

$$\overline{\xi}(\mathbf{p}) := \xi^{\dagger}(\mathbf{p})\gamma^{0}\,. \tag{3.16}$$

With respect to the Dirac dual, the *Elko* have an imaginary bi-orthogonal norm as was already noted in [40, 41]. For the sake of a ready reference, this is recorded explicitly in appendix B.1. The imaginary norm of *Elko* is a hindrance to physical interpretation and quantization. Enormous simplification of interpretation and calculation occurs if we define a new dual with respect to which *Elko* have a real norm. The new dual must have the property that: (a) it yields an invariant real definite norm, and (b) in addition, it must must secure a positive-definite norm for two of the four *Elko*'s, and negative-definite norm for the remaining two. Any other choice will introduce an unjustified element of asymmetry. Up to a relative sign, a unique definition of such a dual, which we call *Elko dual,* is

$$\lambda^{\mathrm{S}}(\mathbf{p}): \qquad \overset{\neg\mathrm{S}}{\lambda}_{\pm,\mp}(\mathbf{p}) := +\left[\rho^{\mathrm{A}}_{\mp,\pm}(\mathbf{p})\right]^{\dagger}\gamma^{0}\,, \tag{3.17}$$

$$\lambda^{\mathrm{A}}(\mathbf{p}): \qquad \overset{\neg\mathrm{A}}{\lambda}_{\pm,\mp}(\mathbf{p}) := -\left[\rho^{\mathrm{S}}_{\mp\pm}(\mathbf{p})\right]^{\dagger}\gamma^{0}\,, \tag{3.18}$$

where the $\rho(\mathbf{p})$ are given in appendix B.2.

The *Elko* dual can also be expressed in the following equivalent, but very useful, form:

$$\text{Elko Dual:} \qquad \overset{\neg}{\lambda}_{\alpha}(\mathbf{p}) := \mathrm{i}\,\varepsilon^{\beta}_{\alpha}\,\lambda^{\dagger}_{\beta}(\mathbf{p})\,\gamma^{0}\,, \tag{3.19}$$

with the antisymmetric symbol $\varepsilon^{\{-,+\}}_{\{+,-\}} := -1 = -\varepsilon^{\{+,-\}}_{\{-,+\}}$. The upper and lower position of indices has been chosen only to avoid expressions like $\varepsilon_{\{+,-\}\{-,+\}}$ and not to imply the use of a metric to raise and lower indices. Equation (3.19) holds for self-conjugate as well as anti-self-conjugate $\lambda(\mathbf{p})$. The Dirac dual, for comparison, may then be re-expressed in the following equivalent form:

$$\text{Dirac Dual:} \qquad \overline{\psi}_{h}(\mathbf{p}) := \delta^{h'}_{h}\,\psi^{\dagger}_{h'}(\mathbf{p})\,\gamma^{0}\,, \tag{3.20}$$

where $\psi(\mathbf{p})$ represents any of the four Dirac spinors and $\delta^{h'}_{h}$ is the Kronecker symbol. Explicitly, equation (3.19) yields

$$\overset{\neg\mathrm{S/A}}{\lambda}_{\{-,+\}}(\mathbf{p}) = +\,\mathrm{i}\,\left[\lambda^{\mathrm{S/A}}_{\{+,-\}}(\mathbf{p})\right]^{\dagger}\gamma^{0}\,, \tag{3.21}$$

$$\overset{\neg\mathrm{S/A}}{\lambda}_{\{+,-\}}(\mathbf{p}) = -\,\mathrm{i}\,\left[\lambda^{\mathrm{S/A}}_{\{-,+\}}(\mathbf{p})\right]^{\dagger}\gamma^{0}\,, \tag{3.22}$$

which, on use of results given in appendix B.2, shows these to be equivalent to definitions (3.17) and (3.18). We have belaboured this point as different expression are useful in various contexts.

3.5 Orthonormality and completeness relations for Elko

With the *Elko* dual thus defined, we now have (by construction)

$$\overset{\neg\mathrm{S}}{\lambda}_{\alpha}(\mathbf{p})\,\lambda^{\mathrm{S}}_{\alpha'}(\mathbf{p}) = +\,2m\,\delta_{\alpha\alpha'}\,, \tag{3.23}$$

$$\overset{\neg\mathrm{A}}{\lambda}_{\alpha}(\mathbf{p})\,\lambda^{\mathrm{A}}_{\alpha'}(\mathbf{p}) = -\,2m\,\delta_{\alpha\alpha'}\,. \tag{3.24}$$

The subscript α ranges over two possibilities: $\{+,-\},\{-,+\}$. The completeness relation

$$\frac{1}{2m}\sum_{\alpha}\left[\lambda^{\rm S}_\alpha(\mathbf{p})\,\overset{\neg{\rm S}}{\lambda}_\alpha(\mathbf{p})-\lambda^{\rm A}_\alpha(\mathbf{p})\,\overset{\neg{\rm A}}{\lambda}_\alpha(\mathbf{p})\right]=\mathbb{I}\,, \tag{3.25}$$

clearly shows the necessity of the anti-self-conjugate spinors. Equations (3.23)–(3.25) have their direct counterpart in Dirac's construct:

$$\overline{u}_h(\mathbf{p})\,u_{h'}(\mathbf{p})=+\,2m\;\delta_{hh'}\,, \tag{3.26}$$

$$\overline{v}_h(\mathbf{p})\,v_{h'}(\mathbf{p})=-\,2m\;\delta_{hh'}\,, \tag{3.27}$$

and

$$\frac{1}{2m}\sum_{h=\pm1/2}\Big[u_h(\mathbf{p})\overline{u}_h(\mathbf{p})-v_h(\mathbf{p})\overline{v}_h(\mathbf{p})\Big]=\mathbb{I}\,. \tag{3.28}$$

4. Establishing $(\mathbf{CPT})^2=-\mathbb{I}$ for Elko

In this section we present the detailed properties of *Elko* spinors under the operation of spatial parity. This prepares us to show that the square of the combined operation of charge conjugation, spatial parity, and time-reversal operators, when acting upon the *Elko*, meets the expectations of Wigner.

4.1 Commutativity of C and P, and parity asymmetry

To set the stage for this section we begin by quoting the unedited textbook wisdom [77]:

$$\left\{\begin{array}{l}\text{bosons: particle and antiparticle have same parity}\\ \text{fermions: particle and antiparticle have opposite parity.}\end{array}\right\} \tag{4.1}$$

To our knowledge the only textbook which tells a more intricate story is that by Weinberg [8]. The only known explicit construct of a theory which challenges the conventional wisdom was reported only about a decade ago in 1993 [20]. In that pure spin-1 bosonic theory particles and antiparticles carry opposite, rather than same, relative intrinsic parity. It manifests itself through the anticommutativity, as opposed to the commutativity, of the $(1,0)\oplus(0,1)$-space's charge conjugation and parity operators. In a somewhat parallel fashion we shall now show that for the spin-$\frac{1}{2}$ *Elko* the charge conjugation operator and parity operator commute, rather than anticommute (as they do for the Dirac case). We shall have more to say about these matters in the concluding section where we bring to our reader's attention the classic work of Wigner [6], and that of Lee and Wick [7].

Given these remarks it does not come as a surprise that the parity operation is slightly subtle for *Elko*. In the $(1/2,0)\oplus(0,1/2)$ representation space it reads

$$P=\mathrm{e}^{\mathrm{i}\Phi}\gamma^0\mathcal{R}\,. \tag{4.2}$$

With $\mathbf{p}:=p\ (\sin(\theta)\cos(\phi),\sin(\theta)\sin(\phi),\cos(\theta))$, the $\mathcal{R}$ reads

$$\mathcal{R}\equiv\{\theta\rightarrow\pi-\theta,\ \phi\rightarrow\phi+\pi,\ p\rightarrow p\}\ . \tag{4.3}$$

This has the consequence that eigenvalues, h, of the helicity operator $\boldsymbol{\sigma} \cdot \widehat{\mathbf{p}}/2$ change sign under the operation of $\mathcal{R}$:

$$\mathcal{R} : h \to h' = -h\,. \tag{4.4}$$

Furthermore, while acting on the Dirac spinors,

$$P u_h(\mathbf{p}) = \mathrm{e}^{\mathrm{i}\Phi}\gamma^0 \mathcal{R} u_h(\mathbf{p}) = \mathrm{e}^{\mathrm{i}\Phi}\gamma^0 u_{-h}(-\mathbf{p}) = -\mathrm{i}\mathrm{e}^{\mathrm{i}\Phi} u_h(\mathbf{p})\,. \tag{4.5}$$

Similarly,

$$P v_h(\mathbf{p}) = \mathrm{i}\mathrm{e}^{\mathrm{i}\Phi} v_h(\mathbf{p})\,. \tag{4.6}$$

Because for the theory based upon Dirac spinors relative intrinsic parity is an observable, we must require the eigenvalues of P to be real. This fixes the phase

$$\mathrm{e}^{\mathrm{i}\Phi} = \pm \mathrm{i}\,. \tag{4.7}$$

The remaining ambiguity, as contained in the sign, still remains. This ambiguity does not affect the physical consequences. It is fixed by recourse to text-book convention by taking the sign on the right-hand side of equation (4.7) to be positive. The parity operator is therefore fixed to be

$$P = \mathrm{i}\gamma^0 \mathcal{R}\,. \tag{4.8}$$

Thus

$$P u_h(\mathbf{p}) = +\, u_h(\mathbf{p})\,, \tag{4.9}$$
$$P v_h(\mathbf{p}) = -\, v_h(\mathbf{p})\,. \tag{4.10}$$

That is, Dirac spinors are eigenspinors of the parity operator. Equations (4.9) and (4.10) imply

$$\textsc{Dirac Spinors}: \quad P^2 = \mathbb{I}\,, \quad [\text{cf equation (4.18)}]\,. \tag{4.11}$$

To calculate the anticommutator, $\{C, P\}$, when acting on the $u_h(\mathbf{p})$ and $v_h(\mathbf{p})$ we now need, in addition, the action of C on these spinors. This action can be summarized as follows:

$$C: \quad \begin{cases} u_{+1/2}(\mathbf{p}) \to -v_{-1/2}(\mathbf{p})\,, & u_{-1/2}(\mathbf{p}) \to v_{+1/2}(\mathbf{p})\,, \\ v_{+1/2}(\mathbf{p}) \to u_{-1/2}(\mathbf{p})\,, & v_{-1/2}(\mathbf{p}) \to -u_{+1/2}(\mathbf{p})\,. \end{cases} \tag{4.12}$$

Using equations (4.9), (4.10), and (4.12) one can readily obtain the action of the anticommutator, $\{C, P\}$, on the four $u(\mathbf{p})$ and $v(\mathbf{p})$ spinors. For each case it is found to vanish:

$$\textsc{Dirac Spinors}: \quad \{C, P\} = 0\,, \quad [\text{cf equation (4.16)}]\,. \tag{4.13}$$

The P acting on the *Elko* yields the result

$$P\lambda^{\mathrm{S}}_{\{-,+\}}(\mathbf{p}) = +\,\mathrm{i}\,\lambda^{\mathrm{A}}_{\{+,-\}}(\mathbf{p})\,, \qquad P\lambda^{\mathrm{S}}_{\{+,-\}}(\mathbf{p}) = -\,\mathrm{i}\,\lambda^{\mathrm{A}}_{\{-,+\}}(\mathbf{p})\,, \tag{4.14}$$

$$P\lambda^{\mathrm{A}}_{\{-,+\}}(\mathbf{p}) = -\,\mathrm{i}\,\lambda^{\mathrm{S}}_{\{+,-\}}(\mathbf{p})\,, \qquad P\lambda^{\mathrm{A}}_{\{+,-\}}(\mathbf{p}) = +\,\mathrm{i}\,\lambda^{\mathrm{S}}_{\{-,+\}}(\mathbf{p})\,. \tag{4.15}$$

That is, *Elko* are *not* eigenspinors of the parity operator. Following the same procedure as before, we now use (4.14), (4.15), and (3.5) — taking a special note of equation (2.11) — to evaluate the action of the commutator $[C, P]$ on each of the four *Elkos*. We find that it vanishes for each of them:

$$\text{ELKO}: \qquad [C, P] = 0\,, \quad [\text{cf equation (4.13)}]\,. \tag{4.16}$$

The commutativity and anticommutativity of the C and P operators is an important distinction between the Dirac spinors and the *Elko*. In this aspect, our results coincide with the possibilities offered by Wigner's general analysis [6]. Despite similarities, our construct differs from the Wigner–Weinberg analysis in a crucial aspect. We outline this in section 12.1. Yet, this difference does not seem to affect many of the general conclusions. Even though a full formal generalization of the Wigner–Weinberg analysis may be desirable, our specific construct does not require it.

Unlike the Dirac spinors, as already noted, equations (4.14) and (4.15) reveal that *Elko* are not eigenstates of P. Furthermore, an apparently paradoxical asymmetry is contained in these equations. For instance, the second equation in (4.14) reads

$$P\lambda^{\mathrm{S}}_{\{+,-\}}(\mathbf{p}) = -\,\mathrm{i}\,\lambda^{\mathrm{A}}_{\{-,+\}}(\mathbf{p})\,. \tag{4.17}$$

As a consequence of (3.13) and (3.14), in the massless/high-energy limit the P-reflection of $\lambda^{\mathrm{S}}_{\{+,-\}}(\mathbf{p})$ identically vanishes. The same happens to the $\lambda^{\mathrm{A}}_{\{+,-\}}(\mathbf{p})$ spinors under P-reflection. This situation is in sharp contrast to the charged-particle spinors. The origin of the asymmetry under P-reflection resides in the fact that the *Elko*, in being dual-helicity objects, combine Weyl spinors of *opposite* helicities. However, in the massless limit, the structures of $\kappa^{(1/2,0)}$ and $\kappa^{(0,1/2)}$ force only positive-helicity $(1/2, 0)$-Weyl and negative-helicity $(0, 1/2)$-Weyl spinors to be non-vanishing. For this reason, in the massless limit the *Elko*, $\lambda^{\mathrm{S}}_{\{-,+\}}(\mathbf{p})$ and $\lambda^{\mathrm{A}}_{\{-,+\}}(\mathbf{p})$, carrying negative-helicity $(1/2, 0)$-Weyl and positive-helicity $(0, 1/2)$-Weyl spinors identically vanish.

Furthermore, the consistency of equations (4.14) and (4.15) requires $P^2 = -\mathbb{I}$ and in the process shows that the remaining two, i.e., the first and the third equation in that set, do not contain additional physical content:

$$\text{ELKO}: \quad P^2 = -\,\mathbb{I}\,. \qquad [\text{cf equation (4.11)}]\,. \tag{4.18}$$

The $(1/2, 0) \oplus (0, 1/2)$ is a P covariant representation space. Yet, in the *Elko* formalism, it carries P-reflection asymmetry. This result has a similar precedence in the Velo-Zwanziger observation, who noted [78] 'the main lesson to be drawn from our analysis is that special relativity is not automatically satisfied by writing equations which transform covariantly'.

4.2 Agreement with Wigner: $(\mathbf{CPT})^{\mathbf{2}} = -\mathbb{I}$

The time-reversal operator $T = \mathrm{i}\gamma^5 C$ acts on *Elko* as follows:

$$T\lambda^{\mathrm{S}}_\alpha(\mathbf{p}) = -\mathrm{i}\lambda^{\mathrm{A}}_\alpha(\mathbf{p}), \qquad T\lambda^{\mathrm{A}}_\alpha(\mathbf{p}) = +\mathrm{i}\lambda^{\mathrm{S}}_\alpha(\mathbf{p})\,, \tag{4.19}$$

implying $T^2 = -\mathbb{I}$. With the action of all three of the C, P and T on *Elko* now known, one can immediately deduce that, in addition to (4.16), we have

$$\textsc{Elko}: \quad [C,T] = 0\,, \quad \{P,T\} = 0\,, \tag{4.20}$$

and that at the same time,

$$(CPT)^2 = -\mathbb{I}\,, \tag{4.21}$$

thus confirming Wigner's expectation. For a discussion of differences with Weinberg, we refer the reader to section 12.1.

5. Spacetime evolution

The existing techniques to specify spacetime evolution do not fully suffice for *Elko*. The path we take carries its inspiration from standard quantum field theory [8, 32], but in the end we had to develop much of the formalism ourselves. So, what follows constitutes in large part our *ab initio* effort.

Section 5.1 establishes that massive *Elko* do not satisfy the Dirac equation. The next subsection, i.e., section 5.2, briefly reflects on the connection between 'spin sums', wave operators, and propagators. The remaining three subsections are devoted to establishing a contrast between *Elko* and Dirac spinors. This exercise not only gives a sharper independent existence to *Elko* but it also sheds new light on the well known Dirac spinors.

5.1 Massive Elko do not satisfy the Dirac equation

For the task at hand it is helpful to make the following local change in notation:

$$\textsc{For Dirac Spinors}: \qquad u_+(\mathbf{p}) \to d_1, \quad u_-(\mathbf{p}) \to d_2, \quad v_+(\mathbf{p}) \to d_3, \quad v_-(\mathbf{p}) \to d_4\,. \tag{5.1}$$

$$\textsc{For Elko}: \qquad \lambda^{\mathrm{S}}_{\{-,+\}}(\mathbf{p}) \to e_1, \quad \lambda^{\mathrm{S}}_{\{+,-\}}(\mathbf{p}) \to e_2, \quad \lambda^{\mathrm{A}}_{\{-,+\}}(\mathbf{p}) \to e_3, \quad \lambda^{\mathrm{A}}_{\{+,-\}}(\mathbf{p}) \to e_4. \tag{5.2}$$

Adopting the procedure introduced in [79], the *Elko* can now be written as

$$e_i = \sum_{j=1}^{4} \Omega_{ij} d_j\,, \qquad i = 1,2,3,4\,, \tag{5.3}$$

where

$$\Omega_{ij} = \begin{cases} +\,(1/2m)\,\overline{d}_j\, e_i \mathbb{I}\,, & \text{for } j = 1,2\,, \\ -\,(1/2m)\,\overline{d}_j\, e_i \mathbb{I}\,, & \text{for } j = 3,4\,. \end{cases} \tag{5.4}$$

In matrix form, Ω reads

$$\Omega = \frac{1}{2}\begin{pmatrix} \mathbb{I} & -\mathrm{i}\mathbb{I} & -\mathbb{I} & -\mathrm{i}\mathbb{I} \\ \mathrm{i}\mathbb{I} & \mathbb{I} & \mathrm{i}\mathbb{I} & -\mathbb{I} \\ \mathbb{I} & \mathrm{i}\mathbb{I} & -\mathbb{I} & \mathrm{i}\mathbb{I} \\ -\mathrm{i}\mathbb{I} & \mathbb{I} & -\mathrm{i}\mathbb{I} & -\mathbb{I} \end{pmatrix} . \tag{5.5}$$

With the definition $\mathcal{B} := (\mathbb{I} + \sigma_2)$, equation (5.5) can be recast into the form[16]

$$\Omega = \frac{1}{2}\begin{pmatrix} \mathcal{B} & -\mathcal{B}^* \\ \mathcal{B}^* & -\mathcal{B} \end{pmatrix} \otimes \mathbb{I} . \tag{5.6}$$

Equations (5.3) and (5.5) immediately tell us that each of the spinors in the set defined by *Elko* is a linear combination of the Dirac *particle and antiparticle* spinors. In momentum space, the Dirac spinors are annihilated by $(\gamma^\mu p_\mu \pm m\mathbb{I})$

$$\begin{cases} \text{For particles:} \qquad (\gamma^\mu p_\mu - m\mathbb{I})\, u(\mathbf{p}) = 0 \,, \\ \text{For antiparticles:} \qquad (\gamma^\mu p_\mu + m\mathbb{I})\, v(\mathbf{p}) = 0 \,. \end{cases} \tag{5.7}$$

That is, Dirac's $u(\mathbf{p})$ and $v(\mathbf{p})$ are eigenspinors of the $\gamma^\mu p_\mu$ operator with eigenvalues $+m$ and $-m$, respectively — a fact emphasized by Weinberg (see, page 225 of [8]) with the observation that it is a result of how the $(1/2, 0)$ and $(0, 1/2)$ representation spaces have been put together to carry simple properties under spatial reflection. Since the mass terms carry opposite signs, and hence are different for the particle and antiparticle, the *Elko* cannot be annihilated by $(\gamma^\mu p_\mu - m\mathbb{I})$, nor by $(\gamma^\mu p_\mu + m\mathbb{I})$. That is, they cannot be eigenspinors of the the $\gamma^\mu p_\mu$ operator. We shall make this result more precise below. Moreover, since the time evolution of the of $u(\mathbf{p})$ occurs via $\exp(-\mathrm{i}p_\mu x^\mu)$ while that for $v(\mathbf{p})$ spinors occurs via $\exp(+\mathrm{i}p_\mu x^\mu)$, one cannot naively go from momentum–space expression (5.3) to its configuration space counterpart.

For formal simplification, we introduce

$$e := \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} , \qquad d := \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix} , \tag{5.8}$$

and

$$\Gamma := \mathbb{I} \otimes \gamma_\mu p^\mu \,. \tag{5.9}$$

In this language, equation (5.3) becomes $e = \Omega d$. Applying from the left the operator Γ and using $[\Gamma, \Omega] = 0$ yields

$$\Gamma e = \Omega \Gamma d \,. \tag{5.10}$$

[16] In what follows the second entry in the direct product always refers to the spinorial part, while the first one refers to the 'ij' part as implied by (5.3).

But, equations (5.7) imply $\Gamma d = m\,\gamma^5 \otimes \mathbb{I}\,d$. Therefore, on using $d = \Omega^{-1}e$, we obtain

$$\Gamma e = \Omega\left(m\,\gamma^5 \otimes \mathbb{I}\right)\Omega^{-1}e\,. \tag{5.11}$$

An explicit evaluation of $\mu := \Omega\left(m\,\gamma^5 \otimes \mathbb{I}\right)\Omega^{-1}$ reveals

$$\mu = m\begin{pmatrix} \sigma_2 & \mathbb{O} \\ \mathbb{O} & -\sigma_2 \end{pmatrix} \otimes \mathbb{I}\,. \tag{5.12}$$

Thus, making the direct product explicit again, finally we reach the result

$$\begin{pmatrix} \gamma_\mu p^\mu & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \gamma_\mu p^\mu & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \gamma_\mu p^\mu & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & \gamma_\mu p^\mu \end{pmatrix}\begin{pmatrix} \lambda^{\mathrm{S}}_{\{-,+\}}(\mathbf{p}) \\ \lambda^{\mathrm{S}}_{\{+,-\}}(\mathbf{p}) \\ \lambda^{\mathrm{A}}_{\{-,+\}}(\mathbf{p}) \\ \lambda^{\mathrm{A}}_{\{+,-\}}(\mathbf{p}) \end{pmatrix} - \mathrm{i}m\mathbb{I}\begin{pmatrix} -\lambda^{\mathrm{S}}_{\{+,-\}}(\mathbf{p}) \\ \lambda^{\mathrm{S}}_{\{-,+\}}(\mathbf{p}) \\ \lambda^{\mathrm{A}}_{\{+,-\}}(\mathbf{p}) \\ -\lambda^{\mathrm{A}}_{\{-,+\}}(\mathbf{p}) \end{pmatrix} = 0\,, \tag{5.13}$$

which establishes that $(\gamma^\mu p_\mu \pm m\mathbb{I})$ do not annihilate the neutral particle spinors[17]. On recalling section 3.4's antisymmetric symbol defined as $\varepsilon^{\{-,+\}}_{\{+,-\}} := -1$, the above equation reduces to

$$\left(\gamma_\mu p^\mu \delta^\beta_\alpha + \mathrm{i}m\mathbb{I}\varepsilon^\beta_\alpha\right)\lambda^{\mathrm{S}}_\beta(\mathbf{p}) = 0\,, \tag{5.14}$$

$$\left(\gamma_\mu p^\mu \delta^\beta_\alpha - \mathrm{i}m\mathbb{I}\varepsilon^\beta_\alpha\right)\lambda^{\mathrm{A}}_\beta(\mathbf{p}) = 0\,. \tag{5.15}$$

These are counterparts of equations (5.7). The presence of δ^β_α in the $\gamma_\mu p^\mu$ term, and the existence of the ε^β_α in the mass term, now make it impossible for the *Elko* to be eigenspinors of the $\gamma_\mu p^\mu$ operator, thus making precise the observation made above. To obtain the configuration-space evolution, we make the standard substitution $p^\mu \rightarrow \mathrm{i}\partial^\mu$, and define

$$\lambda^{\mathrm{S/A}}(x) := \lambda^{\mathrm{S/A}}(\mathbf{p})\ \exp\left(\epsilon^{\mathrm{S/A}} \times \mathrm{i}p_\mu x^\mu\right)\,. \tag{5.16}$$

Consistency with equations (5.14) and (5.15) determines $\epsilon^{\mathrm{S}} = -1$ and $\epsilon^{\mathrm{A}} = +1$, while yielding

$$\left(\mathrm{i}\gamma_\mu \partial^\mu \delta^\beta_\alpha + \mathrm{i}m\mathbb{I}\varepsilon^\beta_\alpha\right)\lambda^{\mathrm{S/A}}_\beta(x) = 0\,. \tag{5.17}$$

Its counterpart for the Dirac case, equation (2.8), has several formal similarities and differences:

1. The Dirac operator, $\mathrm{i}\gamma^\mu\partial_\mu - m\mathbb{I}$, annihilates each of the four $u_h(x)$ and $v_h(x)$. It is not so for the wave operator for *Elko*. The wave operator in equation (5.17) couples the $\{-,+\}$ degree of freedom with the $\{+,-\}$, and vice versa. This is true for self-conjugate as well as anti-self-conjugate *Elko*. Equation (5.17) asks for a eight-component formalism [82, 83] in exactly the same manner as the coupled equations

[17] The result contained in the above equation confirms earlier result of [80] and [81].

for the right-handed Weyl and left-handed Weyl spinors – each of which is a two-component spinor – yields the wave equation for the four-component Dirac spinor. As we proceed, we shall see that the eight-component formalism is not called for as it introduces eight independent degrees of freedom for an inherently four-dimensional representation space.

2. The off-diagonal nature of the mass term in (5.17) is different from a phenomenological off-diagonal Majorana mass term which is often introduced in the context of Dirac equation. This is so because of the observation on the nature of wave operator for *Elko* just enumerated.

3. The Dirac operator can be considered as a 'square root of the Klein–Gordon operator' – often, in introductory lectures, it is even constructed in that way – in the sense that $(\gamma_\mu p^\mu - m\mathbb{I})(\gamma_\mu p^\mu + m\mathbb{I}) = (p_\mu p^\mu - m^2)\mathbb{I}$. This feature remains true for *Elko*: $(\gamma_\mu p^\mu \delta_\alpha^\beta + \mathrm{i}m\mathbb{I}\varepsilon_\alpha^\beta)(\gamma_\mu p^\mu \delta_\alpha^\beta - \mathrm{i}m\mathbb{I}\varepsilon_\alpha^\beta) = (p_\mu p^\mu - m^2)\mathbb{I}\,\delta_\alpha^\beta$. Thus, both Dirac and *Elko* particles have to fulfill the Klein–Gordon equation. Turning this argument around, it is possible to provide a 'quick and dirty derivation' of the wave equation: we would like to consider different square roots of the Klein–Gordon operator times a two-dimensional Kronecker δ of 'helicities' (or referring to (anti-)self-dual spinors in the neutral case), i.e., of $(p_\mu p^\mu - m^2)\mathbb{I}\,\delta_\alpha^\beta$. It is well known how to take the 'square-root' of $p_\mu p^\mu \mathbb{I}$: it yields $\gamma_\mu p^\mu$. Thus, we only have to choose which root of the Kronecker symbol we take. Since its roots are always invertible, the wave equation can always be brought into a form where the first term reads $\gamma_\mu p^\mu\,\delta_\alpha^\beta$, so only the mass term has to be considered. Taking the trivial root, $\pm\delta_\alpha^\beta$, yields the Dirac equation. Taking $(\sigma_2)_\alpha^\beta = \pm\mathrm{i}\varepsilon_\alpha^\beta$ instead produces the wave equations derived above, (5.14) and (5.15). The other Pauli matrices, $\pm\sigma_1$ and $\pm\sigma_3$, i.e., the trace-free symmetric ones, are also possible roots, but they are not considered here.

Before attending to quantum field theoretic structure of the theory we need to collect together some essential new details about the spin sums, and further develop the formalism at the representation space level.

5.2 When wave operators and spin sums do not coincide: a pivotal observation

We draw our reader's attention to the fact that, using the results of appendix B.4, the spin sum over self-conjugate spinors reads

$$\sum_\alpha \lambda_\alpha^{\mathrm{S}}(\mathbf{p}) \stackrel{\neg\mathrm{S}}{\lambda}_\alpha (\mathbf{p}) = m \begin{pmatrix} \mathbb{I} & \mathcal{A}^{\mathrm{S}} \\ \mathcal{A}^{\mathrm{S}} & \mathbb{I} \end{pmatrix}, \tag{5.18}$$

and the one for the anti-self-conjugate spinors is given by

$$\sum_\alpha \lambda_\alpha^{\mathrm{A}}(\mathbf{p}) \stackrel{\neg\mathrm{A}}{\lambda}_\alpha (\mathbf{p}) = m \begin{pmatrix} -\mathbb{I} & \mathcal{A}^{\mathrm{S}} \\ \mathcal{A}^{\mathrm{S}} & -\mathbb{I} \end{pmatrix}. \tag{5.19}$$

The matrix $\mathcal{A}^{\mathrm{S}}$ defines the phase relationship between the $(1/2, 0)$ and $(0, 1/2)$ transforming components of the $\lambda(\mathbf{p})$ spinors. Its explicit form will be obtained in the next section. For

the moment, note may be taken that these spin sums reproduce the completeness relation (3.25), and that in the representation in which the four energy–momentum vector is given by, $p^\mu := (E,\, p\sin(\theta)\cos(\phi),\, p\sin(\theta)\sin(\phi),\, p\cos(\theta))$, $\mathcal{A}^{\mathrm{S}}$ reads

$$\mathcal{A}^{\mathrm{S}} = \begin{pmatrix} 0 & \lambda^* \\ \lambda & 0 \end{pmatrix} , \tag{5.20}$$

where $\lambda := \mathrm{i}e^{\mathrm{i}\phi}$.

We now make what is one of the pivotal observations for the theory. It affects the entire particle interpretation, the realized statistics, the propagator, and the locality structure: *The right-hand sides of the spin sums are not proportional, or unitarily connected, to the momentum–space wave operators in equations (5.14) and (5.15).* This structure contrasts sharply with the Dirac case where the spin sum over the particle spinors

$$\sum_{h=\pm(1/2)} u_h(\mathbf{p})\overline{u}_h(\mathbf{p}) = \gamma_\mu p^\mu + m\mathbb{I} , \tag{5.21}$$

and the one for the antiparticle spinors

$$\sum_{h=\pm(1/2)} v_h(\mathbf{p})\overline{u}_h(\mathbf{p}) = \gamma_\mu p^\mu - m\mathbb{I} , \tag{5.22}$$

correspond to the momentum–space wave operators for the Dirac spinors.

To realize the importance of these contrasting behaviours, the reader may recall that spin sums enter at a profound level in the locality and statistics structure of the theory. So, with these observation in mind, it is important to decipher the origins of spin sums and their relation to wave operators. This we do next on our way to developing the particle interpretation.

5.3 Non-trivial connection between the spin sums and wave operators: introducing $\mathcal{O}$

The question which is now posed is: is there an additional operator which annihilates the $\lambda(\mathbf{p})$ and is different from the ones in (5.14) and (5.15). Such an operator is required to have the property that it does not couple one of the $\lambda(\mathbf{p})$ with the other; and that it annihilates these $\lambda(\mathbf{p})$ singly. Furthermore, such an operator is expected to shed light on the structure of the spin sums which appear in equations (5.18) and (5.19). The meaning of these statements will become more clear as we proceed.

This section is devoted to establishing the existence of such operators, and to reveal their origin and associated properties. We present a unified method which applies not only to the *Elko* but equally well to the Dirac framework. The method is a generalization of the textbook procedure to obtain 'wave operators' [32] with corrections noted in [20,59,60,72]. We introduce a general $(1/2,0)\oplus(0,1/2)$ spinor,

$$\xi(\mathbf{p}) = \begin{pmatrix} \chi^{(1/2,0)}(\mathbf{p}) \\ \chi^{(0,1/2)}(\mathbf{p}) \end{pmatrix} . \tag{5.23}$$

Our task is to obtain the operator(s) defined above. For the Dirac case, it will be found that this operator is nothing but $(\gamma_\mu p^\mu \pm m\mathbb{I})$. For the *Elko* it becomes identical, up to a constant multiplicative factor, to the relevant spin sums. Our walk in search of this operator is leisurely, and we do not refrain from stopping to look at other aspects which these operators may carry. In a particle's rest frame, by definition [32, 59, 60, 72, 84],

$$\chi^{(1/2,0)}(\mathbf{0}) = \mathcal{A}\,\chi^{(0,1/2)}(\mathbf{0})\,. \tag{5.24}$$

Here, the complex 2×2 matrix $\mathcal{A}$ encodes C, P, and T properties of the spinor. It is left unspecified at the moment except that we require it to be invertible. Its most general form may be written, if required, as a general invertible 2×2 matrix times K — where K complex conjugates a spinor to its right. Once $\chi^{(1/2,0)}(\mathbf{0})$ and $\chi^{(0,1/2)}(\mathbf{0})$ are specified the $\chi^{(1/2,0)}(\mathbf{p})$ and $\chi^{(0,1/2)}(\mathbf{p})$ follow from

$$\chi^{(1/2,0)}(\mathbf{p}) = \kappa^{(1/2,0)}\,\chi^{(1/2,0)}(\mathbf{0})\,, \tag{5.25}$$

$$\chi^{(0,1/2)}(\mathbf{p}) = \kappa^{(0,1/2)}\,\chi^{(0,1/2)}(\mathbf{0})\,. \tag{5.26}$$

Equation (5.24) implies

$$\chi^{(0,1/2)}(\mathbf{0}) = \mathcal{A}^{-1}\chi^{(1/2,0)}(\mathbf{0})\,, \tag{5.27}$$

which on immediate use of equation (5.25) yields

$$\chi^{(0,1/2)}(\mathbf{0}) = \mathcal{A}^{-1}\left(\kappa^{(1/2,0)}\right)^{-1}\chi^{(1/2,0)}(\mathbf{p})\,. \tag{5.28}$$

Similarly

$$\chi^{(1/2,0)}(\mathbf{0}) = \mathcal{A}\,\left(\kappa^{(0,1/2)}\right)^{-1}\chi^{(0,1/2)}(\mathbf{p})\,. \tag{5.29}$$

With the useful definition

$$\mathcal{D} := \kappa^{(1/2,0)}\mathcal{A}\,\left(\kappa^{(0,1/2)}\right)^{-1}\,, \tag{5.30}$$

substituting for $\chi^{(1/2,0)}(\mathbf{0})$ from equation (5.29) in equation (5.25), and re-arranging, gives

$$-\,\chi^{(1/2,0)}(\mathbf{p}) \;+\; \mathcal{D}\,\chi^{(0,1/2)}(\mathbf{p}) = 0\,; \tag{5.31}$$

while similar use of equation (5.28) in equation (5.26) results in

$$\mathcal{D}^{-1}\,\chi^{(1/2,0)}(\mathbf{p}) \;-\; \chi^{(0,1/2)}(\mathbf{p}) = 0\,. \tag{5.32}$$

The last two equations, when combined into a matrix form, result in

$$\begin{pmatrix} -\mathbb{I} & \mathcal{D} \\ \mathcal{D}^{-1} & -\mathbb{I} \end{pmatrix}\xi(\mathbf{p}) = 0\,. \tag{5.33}$$

The operator

$$\mathcal{O} := \begin{pmatrix} -\mathbb{I} & \mathcal{D} \\ \mathcal{D}^{-1} & -\mathbb{I} \end{pmatrix}\,, \tag{5.34}$$

which, as we will soon see, is the momentum–space operator we are searching for. We now study its various properties.

5.4 The $\mathcal{O}$ for Dirac spinors

The Dirac representation space is specified by giving $\mathcal{A}$. The $\mathcal{A}$ can be read off from the Dirac rest spinors

$$\mathcal{A} = \begin{cases} +\,\mathbb{I}\,, & \text{for } u(\mathbf{p}) \text{ spinors} \\ -\,\mathbb{I}\,, & \text{for } v(\mathbf{p}) \text{ spinors}\,. \end{cases} \tag{5.35}$$

Parenthetically, we remind the reader that the writing down of the Dirac rest spinors, as shown by Weinberg and also by our independent studies, follows from the following two requirements: (a) parity covariance [8, 72]; and that (b) in a quantum field theoretic framework (with locality imposed), the Dirac field describes fermions [8].

Using information contained in equation (5.35) in equation (5.34), along with the explicit expressions for $\kappa^{(1/2,0)}$ and $\kappa^{(0,1/2)}$, yields

$$\mathcal{O}_{u(\mathbf{p})} = + \begin{pmatrix} -\,\mathbb{I} & \exp\left(\boldsymbol{\sigma}\cdot\boldsymbol{\varphi}\right) \\ \exp\left(-\,\boldsymbol{\sigma}\cdot\boldsymbol{\varphi}\right) & -\,\mathbb{I} \end{pmatrix}, \tag{5.36}$$

$$\mathcal{O}_{v(\mathbf{p})} = - \begin{pmatrix} \mathbb{I} & \exp\left(\boldsymbol{\sigma}\cdot\boldsymbol{\varphi}\right) \\ \exp\left(-\,\boldsymbol{\sigma}\cdot\boldsymbol{\varphi}\right) & \mathbb{I} \end{pmatrix}. \tag{5.37}$$

Exploiting the fact that $(\boldsymbol{\sigma}\cdot\widehat{\mathbf{p}})^2 = \mathbb{I}$, and using the definition of the boost parameter $\boldsymbol{\varphi}$ given in equations (2.4), the exponentials that appear in the above equations take the form

$$\exp\left(\pm\,\boldsymbol{\sigma}\cdot\boldsymbol{\varphi}\right) = \frac{E\mathbb{I} \pm \boldsymbol{\sigma}\cdot\mathbf{p}}{m}\,. \tag{5.38}$$

Using these expansions in equations (5.36) and (5.37), recalling $p_\mu = (E, -\mathbf{p})$, and introducing γ^μ as in equations (2.7), gives equations (5.36) and (5.37) the form

$$\mathcal{O}_{u(\mathbf{p})} = +\,\frac{1}{m}\left(p_\mu\gamma^\mu - m\mathbb{I}\right), \tag{5.39}$$

$$\mathcal{O}_{v(\mathbf{p})} = -\,\frac{1}{m}\left(p_\mu\gamma^\mu + m\mathbb{I}\right). \tag{5.40}$$

Up to a factor of $1/m$, these are the well known momentum–space wave operators for the representation space under consideration. The *linearity* of these operators in p_μ is due to the form of $\mathcal{A}$, and the property of Pauli matrices, $(\boldsymbol{\sigma}\cdot\widehat{\mathbf{p}})^2 = \mathbb{I}$ — see equation (5.38). Comparing equations (5.21) and (5.22) with equations (5.39) and (5.40) results in the following spin sum:

$$\sum_{h=\pm\frac{1}{2}} u_h(\mathbf{p})\overline{u}_h(\mathbf{p}) = -\,m\,\mathcal{O}_{v(\mathbf{p})}\,, \tag{5.41}$$

$$\sum_{h=\pm\frac{1}{2}} v_h(\mathbf{p})\overline{v}_h(\mathbf{p}) = +\,m\,\mathcal{O}_{u(\mathbf{p})}\,. \tag{5.42}$$

This result makes it transparent that $\mathcal{O}$ encodes the spin sums. *As a result, for the Dirac and Majorana fields $\mathcal{O}$ not only determines the wave operator but it also determines the structure of the Feynman–Dyson propagator.* The counterpart of this result for *Elko* will be proved in the next section.

5.5 The $\mathcal{O}$ for Elko

Similarly as above, the *Elko* is specified by giving $\mathcal{A}$. The requirement that the $\lambda(\mathbf{p})$ be dual-helicity eigenspinors of the charge conjugation operator completely determines $\mathcal{A}$ to be

$$\mathcal{A} = \zeta_\lambda \, \Theta \, \beta \,, \tag{5.43}$$

where

$$\beta = \begin{pmatrix} \exp\,(\mathrm{i}\phi) & 0 \\ 0 & \exp\,(-\,\mathrm{i}\phi) \end{pmatrix} , \tag{5.44}$$

and ϕ is the angle defined by the 4-momentum — cf the end of the paragraph above equation (5.20). The result (5.43) is slightly non-trivial but can be extracted from explicit forms of $\lambda(\mathbf{0})$ given in equations (3.9) and (3.10) and by making use of the information in appendix A.1. The given representation accounts for the complex conjugation involved. Explicitly, for the self-conjugate *Elko*, $\mathcal{A}$ is given by

$$\mathcal{A}^{\mathrm{S}} = \begin{pmatrix} 0 & \lambda^* \\ \lambda & 0 \end{pmatrix} , \tag{5.45}$$

where $\lambda := \mathrm{i}e^{\mathrm{i}\phi}$. Note that $\mathcal{A}^{\mathrm{S}}$ is Hermitean and a square root of the unity matrix, i.e., $\mathcal{A}^{\mathrm{S}} = (\mathcal{A}^{\mathrm{S}})^\dagger = (\mathcal{A}^{\mathrm{S}})^{-1}$. For the anti-self-conjugate *Elko*, $\mathcal{A}$ reads

$$\mathcal{A}^{\mathrm{A}} = -\mathcal{A}^{\mathrm{S}} \,. \tag{5.46}$$

In order to obtain $\mathcal{O}$ we must obtain the explicit form of $\mathcal{D}$ for *Elko*. Use of the general procedure implemented for the Dirac representation space gives

$$\mathcal{D} = \pm \frac{E+m}{2m} \left(\mathbb{I} + \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} \right) \mathcal{A}^{\mathrm{S}} \left(\mathbb{I} + \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} \right) , \tag{5.47}$$

where the plus (minus) sign refers to the (anti-) self-conjugate case. Since the anticommutator $\{\boldsymbol{\sigma}\cdot\mathbf{p}, \mathcal{A}_{\mathrm{S}}\}$ vanishes, we have

$$\mathcal{D} = \pm \frac{E+m}{2m} \left(\mathbb{I} + \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} \right) \left(\mathbb{I} - \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} \right) \mathcal{A}^{\mathrm{S}} \,. \tag{5.48}$$

But because $(\boldsymbol{\sigma}\cdot\mathbf{p})^2 = p^2$,

$$\mathcal{D} = \pm \frac{(E+m)^2 - p^2}{2m(E+m)} \mathcal{A}^{\mathrm{S}} = \pm \mathcal{A}^{\mathrm{S}} \,, \tag{5.49}$$

where the last identity is due to the dispersion relation (12.5). Consequently,

$$\mathcal{O}_{\lambda^{\mathrm{S}}(\mathbf{p})} = + \begin{pmatrix} -\,\mathbb{I} & \mathcal{A}^{\mathrm{S}} \\ \mathcal{A}^{\mathrm{S}} & -\,\mathbb{I} \end{pmatrix} , \tag{5.50}$$

$$\mathcal{O}_{\lambda^{\mathrm{A}}(\mathbf{p})} = - \begin{pmatrix} \mathbb{I} & \mathcal{A}^{\mathrm{S}} \\ \mathcal{A}^{\mathrm{S}} & \mathbb{I} \end{pmatrix} . \tag{5.51}$$

Keeping relation (5.46) in mind, the comparison of equations (5.18) and (5.19) with equations (5.50) and (5.51) yields

$$\sum_\alpha \lambda^{\mathrm{S}}_\alpha(\mathbf{p}) \stackrel{\neg\mathrm{S}}{\lambda_\alpha}(\mathbf{p}) = -\, m\, \mathcal{O}_{\lambda^{\mathrm{A}}(\mathbf{p})}\,, \tag{5.52}$$

$$\sum_\alpha \lambda^{\mathrm{A}}_\alpha(\mathbf{p}) \stackrel{\neg\mathrm{A}}{\lambda_\alpha}(\mathbf{p}) = +\, m\, \mathcal{O}_{\lambda^{\mathrm{S}}(\mathbf{p})}\,. \tag{5.53}$$

Thus, as in the Dirac case, it is transparent that in this case as well $\mathcal{O}$ encodes the spin sums. As a result, for the two new fields based upon Elko, $\mathcal{O}$ *does not* determine the wave operator but it *still determines* the structure of the Feynman–Dyson propagator. The part of the assertion which refers to the propagator will be proved below.

For later convenience we introduce the ϕ-dependent matrix

$$\mathcal{G} := \begin{pmatrix} \mathbb{O} & \mathcal{A}^{\mathrm{S}} \\ \mathcal{A}^{S} & \mathbb{O} \end{pmatrix}. \tag{5.54}$$

In terms of $\mathcal{G}$ the spin sums read

$$\sum_\alpha \lambda^{\mathrm{S}}_\alpha(\mathbf{p}) \stackrel{\neg\mathrm{S}}{\lambda_\alpha}(\mathbf{p}) = +m(\mathbb{I}+\mathcal{G})\,, \tag{5.55}$$

$$\sum_\alpha \lambda^{\mathrm{A}}_\alpha(\mathbf{p}) \stackrel{\neg\mathrm{A}}{\lambda_\alpha}(\mathbf{p}) = -m\,(\mathbb{I}-\mathcal{G})\,. \tag{5.56}$$

Some remarks are in order. First, it should be noted that the operators

$$\mathcal{P}^{\pm} := \frac{1}{2}(\mathbb{I}\pm\mathcal{G})\,, \tag{5.57}$$

form a complete set of projection operators. Second, the definition of $\mathcal{G}$ implies the identity

$$\mathcal{G}(\phi) = -\mathcal{G}(\phi+\pi)\,. \tag{5.58}$$

This corresponds to the behaviour of $\mathcal{G}$ in going from $+\,\mathbf{p}$ to $-\,\mathbf{p}$, i.e., to $\mathcal{G}(\mathbf{p}) = -\,\mathcal{G}(-\mathbf{p})$. It is curious that the spin sums for the *Elko* depend only on ϕ, and are independent of θ and of p. This matter is considered in appendix B.6. Furthermore, it is observed that although the spin sums do not coincide with the wave operator they can be still be considered as 'square roots of the Klein–Gordon operator' in the following sense. *If one does not employ any dispersion relation*, from (5.48) it follows that setting the product $\mathcal{P}^+\mathcal{P}^-$ to zero *implies* the dispersion relation (12.5), which is nothing but the Klein–Gordon operator in momentum space. More explicitly, we recall that arriving at results (5.55) and (5.56) has already invoked the dispersion relation. One may, going a step backward, define

$$\mathcal{P}^{\mathrm{S}} = -\frac{1}{2}\mathcal{O}_{\lambda^{\mathrm{A}}(\mathbf{p})} = +\frac{1}{2m}\sum_\alpha \lambda^{\mathrm{S}}_\alpha(\mathbf{p}) \stackrel{\neg\mathrm{S}}{\lambda_\alpha}(\mathbf{p})\,, \tag{5.59}$$

$$\mathcal{P}^{\mathrm{A}} = -\frac{1}{2}\mathcal{O}_{\lambda^{\mathrm{S}}(\mathbf{p})} = -\frac{1}{2m}\sum_\alpha \lambda^{\mathrm{A}}_\alpha(\mathbf{p}) \stackrel{\neg\mathrm{A}}{\lambda_\alpha}(\mathbf{p})\,, \tag{5.60}$$

where the $\mathcal{O}_{\lambda^{\mathrm{S,A}}(\mathbf{p})}$ are now written without invoking the second equality in equation (5.49), but instead are defined using equation (5.48) — the only difference being that now no dispersion relation is invoked explicitly. Then, $\mathcal{P}^{\mathrm{S}}\mathcal{P}^{\mathrm{A}} = 0$ implies the dispersion relation.

Invoking the dispersion relation, as in equation (5.49), makes $\mathcal{P}^{\mathrm{S}}$ and $\mathcal{P}^{\mathrm{A}}$ identical to $\mathcal{P}^{+}$ and $\mathcal{P}^{-}$, respectively. This exercise may, in addition, also be viewed as a simple consistency check.

6. Particle interpretation and mass dimensionality

It will be established in the next section that the quantum field associated with *Elko* is

$$\eta(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\,m\,E(\mathbf{p})}} \sum_{\beta} \left[c_\beta(\mathbf{p}) \lambda^{\mathrm{S}}_\beta(\mathbf{p}) \mathrm{e}^{-\mathrm{i}p_\mu x^\mu} + c^\dagger_\beta(\mathbf{p}) \lambda^{\mathrm{A}}_\beta(\mathbf{p}) \mathrm{e}^{+\mathrm{i}p_\mu x^\mu} \right] , \tag{6.1}$$

with the corresponding *Elko* dual given by

$$\stackrel{\neg}{\eta}(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\,m\,E(\mathbf{p})}} \sum_{\beta} \left[c^\dagger_\beta(\mathbf{p}) \stackrel{\neg S}{\lambda_\beta}(\mathbf{p}) \mathrm{e}^{+\mathrm{i}p_\mu x^\mu} + c_\beta(\mathbf{p}) \stackrel{\neg A}{\lambda_\beta}(\mathbf{p}) \mathrm{e}^{-\mathrm{i}p_\mu x^\mu} \right] . \tag{6.2}$$

Here

$$\left\{ c_\beta(\mathbf{p}),\ c^\dagger_{\beta'}(\mathbf{p}') \right\} = (2\pi)^3\, \delta^3\left(\mathbf{p} - \mathbf{p}'\right) \delta_{\beta\beta'} \,, \tag{6.3}$$

$$\left\{ c^\dagger_\beta(\mathbf{p}),\ c^\dagger_{\beta'}(\mathbf{p}') \right\} = \left\{ c_\beta(\mathbf{p}),\ c_{\beta'}(\mathbf{p}') \right\} = 0 \,. \tag{6.4}$$

6.1 The Elko propagator

Once these fields and anticommutators are given, the amplitude for a positive-energy self-conjugate particle to propagate from x to x' is[18]

$$\langle s(x') | s(x) \rangle \,. \tag{6.5}$$

The state $|s(x)\rangle$ contains 1 positive-energy self-conjugate particle of mass m. From equation (6.2) we decipher that the state which contains 1 positive-energy self-conjugate particle at x is $\stackrel{\neg}{\eta}(x)|\quad\rangle$, where $|\quad\rangle$ represents the physical vacuum. Therefore, the covariant amplitude (6.5) is[19]

$$\mathcal{Q}_{x\to x'} = \varpi \langle \quad | \eta(x') \stackrel{\neg}{\eta}(x) | \quad \rangle \,. \tag{6.6}$$

The pre-factor, $\varpi \in \mathbb{C}$, shall be fixed by the requirement that $\mathcal{Q}_{x\to x'}$, when integrated over all spacetime, yields *unity* (or, to be more precise, $\exp(\mathrm{i}\gamma)$, with $\gamma \in \mathbb{R}$). This requirement is imposed by the quantum-mechanical interpretational structure for such amplitudes: the $\mathcal{Q}_{x\to x'}$ integrated over all spacetime, call it $\mathcal{A}$, yields the amplitude for the particle to be found anywhere in the universe; consequently, $\mathcal{A}^* \mathcal{A}$ is the corresponding probability. For a

[18] This argument follows closely Hatfield's discussion in [85] for the Dirac case.

[19] We call $\langle \quad | \eta(x') \stackrel{\neg}{\eta}(x) | \quad \rangle$ the covariant amplitude, to distinguish it from $\langle \quad | \eta(x') \eta^\dagger(x) | \quad \rangle$, which is often referred to as an amplitude.

free particle this interpretation is free from interpretational problems. For the interacting case, the relation of $\mathcal{Q}_{x\to x'}$ with an appropriate Green function provides the additional interpretational structure. Moreover, the integration over the entire spacetime is required, rather than integration over the part of spacetime which carries the lightcone of x as its (open) boundary, as the classically forbidden region remains accessible quantum mechanically[20]. The Feynman–Dyson propagator is then nothing but a numerical constant times $\mathcal{Q}_{x\to x'}$, where the constant is determined by the requirement that it coincides with the appropriate Green function. The quantum-mechanical propagation of the self-conjugate particle, like the one in case of the Dirac particle, is the process where a positive-energy self-conjugate particle is created out of the vacuum at x, propagates to x', where it is re-absorbed into the vacuum. In a given inertial frame[21] one cannot destroy a particle before its creation; therefore, $t' > t$.

As in the Dirac case, there is a distinct but physically equivalent process that we must also take into account. If we consider positive-energy self-conjugate particles to carry negative charge $-\ell$, $(\ell > 0)$[22] then the process above lowers this charge by one unit at x and subsequently raises it by one unit at x'. Negative-energy self-conjugate particles propagating backward in time, that is, positive-energy anti-self-conjugate particles, carry the opposite charge, ℓ. If we create an anti-self-conjugate particle of mass m at x', transport it to x, where we destroy it, then we are also raising the new charge at x' by one unit and lowering it by the same amount at x. From equation (6.1) and the particle interpretation, we see that $\eta(x)$ creates anti-self-conjugate particles; so the covariant amplitude for this process is

$$\varpi\langle\quad|\overset{\neg}{\eta}(x)\eta(x')|\quad\rangle\,. \tag{6.7}$$

Once again, we cannot destroy a particle before we create it; hence this process is physically meaningful only for $t > t'$. Since fermionic amplitudes are antisymmetric under the exchange $x \leftrightarrow x'$, the total covariant amplitude for the process under consideration is

$$\varpi\langle\quad|\eta(x')\overset{\neg}{\eta}(x)|\quad\rangle\theta(t'-t) - \varpi\langle\quad|\overset{\neg}{\eta}(x)\eta(x')|\quad\rangle\theta(t-t')\,. \tag{6.8}$$

Invoking the fermionic time-ordering operator $\mathcal{T}$ this may be recast as

$$\varpi\langle\quad|\mathcal{T}(\eta(x')\overset{\neg}{\eta}(x))|\quad\rangle\,. \tag{6.9}$$

[20] We are following this first-principle derivation so as to avoid using full quantum field theoretic formalism which implicitly contains the assumption of locality. The latter property, as we shall see, is not fully respected by the *Elko* quantum field.

[21] It is important to make this reference to an inertial frame because quantum mechanically a particle has a finite probability of 'tunneling' beyond the light cone of x. Such a tunneling destroys the time ordering of events. See also footnote 12.

[22] This charge, which is pseudo-scalar under the operation of parity, is distinct from Dirac charge — irrespective of the fact whether it is zero, or non-vanishing. The pseudo-scalar nature of ℓ follows from the fact that the *Elko* lose their self-/anti-self-conjugacy under the transformation $\lambda(x) \to \exp\left[\mathrm{i}\alpha(x)\right]\lambda(x)$; $\alpha \in \mathbb{R}$. It is preserved under the local gauge transformation $\lambda(x) \to \exp\left[\mathrm{i}\alpha(x)\gamma^5\right]\lambda(x)$.

We now evaluate $\langle\ \ |\mathcal{T}(\eta(x')\overset{\neg}{\eta}(x))|\ \ \rangle$ term by term. The first term may be written as

$$\Big\langle\ \ \Big|\int \frac{\mathrm{d}^3p}{(2\pi)^3}\ \ \frac{1}{\sqrt{2\,m\,E(\mathbf{p})}}\int\frac{\mathrm{d}^3p'}{(2\pi)^3}\frac{1}{\sqrt{2\,m\,E(\mathbf{p}')}}$$
$$\times\sum_{\beta'}\sum_{\beta}\Big[c_{\beta'}(\mathbf{p}')\lambda^{\mathrm{S}}_{\beta'}(\mathbf{p}')\mathrm{e}^{-\mathrm{i}p'_\mu x'^\mu}+c^\dagger_{\beta'}(\mathbf{p}')\lambda^{\mathrm{A}}_{\beta'}(\mathbf{p}')\mathrm{e}^{+\mathrm{i}p'_\mu x'^\mu}\Big]$$
$$\times\Big[c^\dagger_\beta(\mathbf{p})\overset{\neg\mathrm{S}}{\lambda_\beta}(\mathbf{p})\mathrm{e}^{+\mathrm{i}p_\mu x^\mu}+c_\beta(\mathbf{p})\overset{\neg\mathrm{A}}{\lambda_\beta}(\mathbf{p}))\mathrm{e}^{+\mathrm{i}p_\mu x^\mu}\Big]\Big|\ \ \Big\rangle\theta(t'-t)\,. \tag{6.10}$$

The only non-vanishing contribution comes from the terms in the expansion of the form $c_{\beta'}(\mathbf{p}')c^\dagger_\beta(\mathbf{p})$. Under the integral, $\langle\ \ |c_{\beta'}(\mathbf{p}')c^\dagger_\beta(\mathbf{p})|\ \ \rangle$, using equation (6.3), may be replaced by $\langle\ \ |-c^\dagger_\beta(\mathbf{p})c_{\beta'}(\mathbf{p}')|\ \ \rangle+(2\pi)^3\delta^3(\mathbf{p}'-\mathbf{p})\delta_{\beta'\beta}$ (with the already made implicit assumption that the vacuum state is normalized to unity). Taking further note of the fact that $\langle\ \ |c^\dagger_\beta(\mathbf{p})c_{\beta'}(\mathbf{p}')|\ \ \rangle$ identically vanishes, this then yields the first term in $\langle\ \ |\mathcal{T}(\eta(x')\overset{\neg}{\eta}(x))|\ \ \rangle$ to be

$$\int\frac{\mathrm{d}^3p}{(2\pi)^3}\ \frac{1}{2mE(\mathbf{p})}\sum_\beta\lambda^{\mathrm{S}}_\beta(\mathbf{p})\overset{\neg\mathrm{S}}{\lambda_\beta}(\mathbf{p})\mathrm{e}^{-\mathrm{i}p_\mu(x'^\mu-x^\mu)}\theta(t'-t)\,. \tag{6.11}$$

A similar evaluation of the second term gives

$$-\int\frac{\mathrm{d}^3p}{(2\pi)^3}\ \frac{1}{2mE(\mathbf{p})}\sum_\beta\lambda^{\mathrm{A}}_\beta(\mathbf{p})\overset{\neg\mathrm{A}}{\lambda_\beta}(\mathbf{p})\mathrm{e}^{+\mathrm{i}p_\mu(x'^\mu-x^\mu)}\theta(t-t')\,. \tag{6.12}$$

Combining both of these evaluations leads to the result

$$\mathcal{Q}_{x\to x'}=\varpi\int\frac{\mathrm{d}^3p}{(2\pi)^3}\ \ \frac{1}{2mE(\mathbf{p})}\sum_\beta\Big[\theta(t'-t)\lambda^{\mathrm{S}}_\beta(\mathbf{p})\overset{\neg\mathrm{S}}{\lambda_\beta}(\mathbf{p})\mathrm{e}^{-\mathrm{i}p_\mu(x'^\mu-x^\mu)}$$
$$-\theta(t-t')\lambda^{\mathrm{A}}_\beta(\mathbf{p})\overset{\neg\mathrm{A}}{\lambda_\beta}(\mathbf{p})\mathrm{e}^{+\mathrm{i}p_\mu(x'^\mu-x^\mu)}\Big]\,. \tag{6.13}$$

For further simplification we invoke the spin sums for $\lambda^{\mathrm{S}}(\mathbf{p})$ and $\lambda^{\mathrm{A}}(\mathbf{p})$ given in equations (5.55) and (5.56):

$$\mathcal{Q}_{x\to x'}=\varpi\int\ \frac{\mathrm{d}^3p}{(2\pi)^3}\ \frac{1}{2E(\mathbf{p})}\Big[\theta(t'-t)(\mathbb{I}+\mathcal{G}(\phi))\,\mathrm{e}^{-\mathrm{i}p_\mu(x'^\mu-x^\mu)}$$
$$+\theta(t-t')(\mathbb{I}-\mathcal{G}(\phi))\,\mathrm{e}^{+\mathrm{i}p_\mu(x'^\mu-x^\mu)}\Big]\,. \tag{6.14}$$

Next focus on the second term. Letting $\mathbf{p}\to-\mathbf{p}$, and using (5.58), we get

$$\mathcal{Q}_{x\to x'}=\varpi\int\ \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E(\mathbf{p})}\Big[\theta(t'-t)(\mathbb{I}+\mathcal{G}(\phi))\,\mathrm{e}^{-\mathrm{i}E(\mathbf{p})(t'-t)\,+\,\mathrm{i}\,\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})}$$
$$+\theta(t-t')(\mathbb{I}+\mathcal{G}(\phi))\,\mathrm{e}^{+\mathrm{i}E(\mathbf{p})(t'-t)+\,\mathrm{i}\,\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})}\Big]\,. \tag{6.15}$$

The above equation can be cast into a covariant form by the use of an integral representation of the Heaviside step function

$$\text{For the first term:}\qquad \theta(t'-t)=\lim_{\epsilon\to0^+}\int\frac{\mathrm{d}\omega}{2\pi\mathrm{i}}\frac{\mathrm{e}^{\mathrm{i}\omega(t'-t)}}{\omega-\mathrm{i}\epsilon}\,, \tag{6.16}$$

$$\text{For the second term:}\qquad \theta(t-t')=\lim_{\epsilon\to0^+}\int\frac{\mathrm{d}\omega}{2\pi\mathrm{i}}\frac{\mathrm{e}^{\mathrm{i}\omega(t-t')}}{\omega-\mathrm{i}\epsilon}\,. \tag{6.17}$$

Inserting these integrals into (6.15) yields

$$\mathcal{Q}_{x\to x'} = -\mathrm{i}\,\varpi \lim_{\epsilon\to 0^+}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E(\mathbf{p})}\int\frac{\mathrm{d}\omega}{2\pi}$$
$$\times\left[\frac{(\mathbb{I}+\mathcal{G}(\phi))}{\omega-\mathrm{i}\epsilon}\left(\mathrm{e}^{\mathrm{i}(\omega-E(\mathbf{p}))(t'-t)}\,\mathrm{e}^{\mathrm{i}\mathbf{p}.(\mathbf{x}'-\mathbf{x})}+\mathrm{e}^{-\mathrm{i}(\omega-E(\mathbf{p}))(t'-t)}\,\mathrm{e}^{\mathrm{i}\mathbf{p}.(\mathbf{x}'-\mathbf{x})}\right)\right]. \tag{6.18}$$

Now we change variables: in the first integral $\omega\to p_0 = -\left(\omega - E(\mathbf{p})\right)$, while in the second integral $\omega\to p_0=\omega-E(\mathbf{p})$. This substitution alters (6.18) to

$$\mathcal{Q}_{x\to x'} = -\mathrm{i}\,\varpi \lim_{\epsilon\to 0^+}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{2E(\mathbf{p})}\mathrm{e}^{-\mathrm{i}p_\mu(x'^\mu-x^\mu)}\left[\frac{\mathbb{I}+\mathcal{G}(\phi)}{E(\mathbf{p})-p_0-\mathrm{i}\epsilon}+\frac{\mathbb{I}+\mathcal{G}(\phi)}{E(\mathbf{p})+p_0-\mathrm{i}\epsilon}\right]. \tag{6.19}$$

Because of the indicated limit on the above integrals, we can drop the terms of the order ϵ^2 and write the result as

$$\mathcal{Q}_{x\to x'} = \int\frac{\mathrm{d}^4p}{(2\pi)^4}\;\mathrm{e}^{-\mathrm{i}p_\mu(x'^\mu-x^\mu)}\;\left[\mathrm{i}\,\varpi\frac{(\mathbb{I}+\mathcal{G}(\phi))}{p_\mu p^\mu-m^2+\mathrm{i}\epsilon}\right], \tag{6.20}$$

where the limit $\epsilon\to 0^+$ is now understood. The covariant amplitude is directly related to the Feynman–Dyson propagator:

$$\mathcal{S}_{\mathrm{FD}}(x',x)\propto\mathcal{Q}_{x\to x'}, \tag{6.21}$$

where the proportionality constant is determined by the requirement that $\mathcal{S}_{\mathrm{FD}}(x',x)$ coincides with the appropriate Green function (see equations (6.27)–(6.29) below).

6.2 Mass dimension one: the Elko propagator in the absence of a preferred direction

If there is no preferred direction, and since we are integrating over all momenta, we are free to choose a coordinate system in which $\mathbf{x}'-\mathbf{x}$ lies along the $\widehat{z}$ direction. In this special case, the $\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})$ depends only on p and θ, but not on ϕ. Thus, the only ϕ-dependence in the whole integrand comes from $\mathcal{G}$ which depends on ϕ in such a manner that an integral over one period vanishes. With this result at hand, the covariant amplitude reduces to

$$\mathcal{Q}_{x\to x'} = \int\frac{\mathrm{d}^4p}{(2\pi)^4}\;\mathrm{e}^{-\mathrm{i}p_\mu(x'^\mu-x^\mu)}\;\left[\mathrm{i}\,\varpi\frac{\mathbb{I}}{p_\mu p^\mu-m^2+\mathrm{i}\epsilon}\right]. \tag{6.22}$$

The $\mathcal{Q}_{x\to x'}$ depends not on x but on $x-x'$. This is consistent with the observation that we have no preferred spacetime point. For this reason we integrate $\mathcal{Q}_{x\to x'}$ over all possible $x-x'$ and set the result to unity[23]:

$$(2\pi)^4\int\frac{\mathrm{d}^4p}{(2\pi)^4}\delta^4(p^\mu)\left[\mathrm{i}\,\varpi\frac{\mathbb{I}}{p_\mu p^\mu-m^2+\mathrm{i}\epsilon}\right]=1 \tag{6.23}$$

[23]To be more precise, one should demand the amplitude to be $\exp(\mathrm{i}\gamma)$, with $\gamma\in\mathbb{R}$.

That is,

$$\mathrm{i}\,\varpi\frac{\mathbb{I}}{-m^2+\mathrm{i}\epsilon}=\mathbb{I}\,. \tag{6.24}$$

Taking the limit $\epsilon\rightarrow 0$ yields

$$\varpi=\mathrm{i}m^2\,. \tag{6.25}$$

With ϖ now fixed we have

$$\mathcal{Q}_{x\rightarrow x'}=-m\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}p_\mu(x'^\mu-x^\mu)}\,\left[\frac{m\,\mathbb{I}}{p_\mu p^\mu-m^2+\mathrm{i}\epsilon}\right]. \tag{6.26}$$

Therefore, the propagator for the new theory is

$$\mathcal{S}_{\mathrm{FD}}^{\mathrm{Elko}}(x',x)\;:=-\,\frac{1}{m^2}\mathcal{Q}_{x\rightarrow x'} \tag{6.27}$$

$$=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}p_\mu(x'^\mu-x^\mu)}\,\left[\frac{\mathbb{I}}{p_\mu p^\mu-m^2+\mathrm{i}\epsilon}\right], \tag{6.28}$$

and it satisfies

$$\left(\partial_{\mu'}\partial^{\mu'}\mathbb{I}+m^2\mathbb{I}\right)\mathcal{S}_{\mathrm{FD}}^{\mathrm{Elko}}(x',x)=-\,\delta^4\left(x'-x\right)\,. \tag{6.29}$$

Thus, it is again clear that the new theory is different from that based upon Dirac spinors. For the latter, the counterpart of equation (6.28) reads

$$\mathcal{S}_{\mathrm{FD}}^{\mathrm{Dirac}}(x',x)=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}p_\mu(x'^\mu-x^\mu)}\,\left[\frac{\gamma_\mu p^\mu+m\mathbb{I}}{p_\mu p^\mu-m^2+\mathrm{i}\epsilon}\right]. \tag{6.30}$$

This is valid for the Majorana as well as the Dirac field.

In the absence of a preferred direction, the Elko propagator is identical to that of a scalar Klein–Gordon field. It is this circumstance that endows the $\eta(x)$ with mass dimension one. General discussion on mass dimensionality of fields and their relation with propagators can be found, for example, in section 12.1 of [8]. For the circumstance at hand it is to be noted that Weinberg's purely dimensional arguments — which appear after equation (12.1.9) of [8] — are more robust than the arguments that precede it. The reason lies in the assumption of locality. For local massive fields, combining his dimensional arguments with those following his monograph's equation (12.1.2), one immediately sees that for a massive field of Lorentz transformation type (A,B), the mass dimension of the field is $1+A+B$. So for the field of the types $(1/2,0)$ and $(0,1/2)$, the expected mass dimensionality follows to be $3/2$. The reason that for the *Elko* field, which also transforms as $(1/2,0)\oplus(0,1/2)$, the mass dimension is not $3/2$, but 1, lies in the non-locality of the field $\eta(x)$, a property we establish in section 8.

A heuristic understanding of this crucial result may be gained as follows. Equation (5.13), or its equivalent equation (5.17), constitutes a set of coupled equations. The set for the self-conjugate *Elko* reads

$$\gamma_\mu p^\mu\lambda^{\mathrm{S}}_{\{-,+\}}(\mathbf{p})+\mathrm{i}m\lambda^{\mathrm{S}}_{\{+,-\}}(\mathbf{p})=0\,, \tag{6.31}$$

$$\gamma_\mu p^\mu\lambda^{\mathrm{S}}_{\{+,-\}}(\mathbf{p})-\mathrm{i}m\lambda^{\mathrm{S}}_{\{-,+\}}(\mathbf{p})=0\,. \tag{6.32}$$

The second of these equations may be re-written as

$$\lambda^{\mathrm{S}}_{\{-,+\}}(\mathbf{p}) = \frac{\gamma_\mu p^\mu}{\mathrm{i}m}\lambda^{\mathrm{S}}_{\{+,-\}}(\mathbf{p})\,. \tag{6.33}$$

Substitution of this in equation (6.31) gives $\left(\gamma_\mu\gamma_\nu p^\mu p^\nu - m^2\right)\lambda^{\mathrm{S}}_{\{+,-\}}(\mathbf{p}) = 0$, which on exploiting the commutator $[p^\mu, p^\nu] = 0$, reads

$$\left(\frac{\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu}{2}p^\mu p^\nu - m^2\mathbb{I}\right)\lambda^{\mathrm{S}}_{\{+,-\}}(\mathbf{p}) = 0\,. \tag{6.34}$$

Using the anticommutator $\{\gamma^\mu, \gamma^\nu\} = 2\eta_{\mu\nu}\mathbb{I}$, equation (6.34) becomes

$$\left(\eta_{\mu\nu}p^\mu p^\nu\mathbb{I} - m^2\mathbb{I}\right)\lambda^{\mathrm{S}}_{\{+,-\}}(\mathbf{p}) = 0\,. \tag{6.35}$$

Similarly, it is seen that $\left(\eta_{\mu\nu}p^\mu p^\nu\mathbb{I} - m^2\mathbb{I}\right)$ annihilates the remaining three $\lambda(\mathbf{p})$. Thus, we have

$$\left(\eta_{\mu\nu}p^\mu p^\nu\mathbb{I} - m^2\mathbb{I}\right)\lambda^{\mathrm{S/A}}(\mathbf{p}) = 0\,, \tag{6.36}$$

which is nothing but the Klein–Gordon equation in momentum space.

For the *Elko* field the crucial difference as compared to the Dirac case is that the ‘square root’ of the Klein–Gordon operator is not a valid wave operator for a *single* $\lambda^{\mathrm{S/A}}_\alpha(\mathbf{p})$, but rather, unless $m = 0$, it *couples* the two four-component degrees of freedom, $\{+,-\}$ and $\{-,+\}$ — see equations (6.31) and (6.32). This has led some authors to suggest an eight-component formalism [82, 83]. However, the spacetime evolution considered above makes it abundantly clear that the Klein–Gordon propagation is an *intrinsic* property of the *Elko* field, and Klein–Gordon operator is the kinetic operator for this field. In other words, a conflict arises between the canonical quantization and the path integral approach unless equation (6.36) is taken to define the Lagrangian density for the Elko framework. This circumstance makes the Elko field to carry mass dimension one, rather than three halves.

The $\mathcal{G}(\mathbf{p})$ independence of the theory must be re-examined if a preferred direction exists. This may come about because of a cosmic preferred direction [86], a fixed background, a thermal bath, a reference fluid, or some other external field such as a magnetic/Thirring–Lense field of a neutron star. In addition, the following interpretational structure is worth taking note of, explicitly.

- The $c^\dagger_{\{\mp,\pm\}}$ creates a positive-energy self-conjugate particle with dual helicity $\{\mp,\pm\}$.

- The $c_{\{\mp,\pm\}}$ destroys a negative-energy self-conjugate particle with dual helicity $\{\mp,\pm\}$. That is, $c_{\{\mp,\pm\}}$ creates a positive-energy hole with the reversed dual helicity $\{\pm,\mp\}$. The holes carry the interpretation of anti-self-conjugate particles. Thus $c_{\{\mp,\pm\}}$ is also the creator of positive-energy anti-self-conjugate particles with the reversed helicity $\{\pm,\mp\}$.

7. Energy of vacuum and establishing the fermionic statistics

The spacetime evolution as contained in equations (5.16) and (5.17), or equivalently in (6.36), allows us to introduce the field operator:

$$\eta(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{a(p^\mu)}{2\,E(\mathbf{p})}\sum_\beta \left[c_\beta(\mathbf{p})\lambda^{\mathrm{S}}_\beta(\mathbf{p})\mathrm{e}^{-\mathrm{i}p_\mu x^\mu} + c^\dagger_\beta(\mathbf{p})\lambda^{\mathrm{A}}_\beta(\mathbf{p})\mathrm{e}^{+\mathrm{i}p_\mu x^\mu}\right], \tag{7.1}$$

where $a(p^\mu) \in \mathbb{R}$. It will be fixed below. Our aim now is to settle the statistics, i.e., commutative/anticommutative properties of the $c_\beta(\mathbf{p})$ and $c^\dagger_\beta(\mathbf{p})$, and to establish the result with which the previous section opened. Towards this task we proceed as follows. The field operator $\eta(x)$ satisfies

$$\left(\mathbb{I}\eta_{\mu\nu}\partial^\mu\partial^\nu + m^2\mathbb{I}\right)\eta(x) = 0\,. \tag{7.2}$$

To obtain the momenta conjugate to $\eta(x)$, we note that equation (7.2) follows from the action

$$S\left[\overset{\neg}{\eta}(x), \eta(x)\right] = \int \mathrm{d}^4 x\, \mathcal{L}\left(\overset{\neg}{\eta}(x), \eta(x)\right) \tag{7.3}$$

$$= \int \mathrm{d}^4 x\, \left(\partial^\mu \overset{\neg}{\eta}(x)\,\partial_\mu \eta(x) - m^2\, \overset{\neg}{\eta}(x)\,\eta(x)\right). \tag{7.4}$$

The field momentum conjugate to $\eta(x)$ is

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\,\dot{\eta}} = \frac{\partial}{\partial t}\overset{\neg}{\eta}(x)\,, \tag{7.5}$$

while the Hamiltonian density reads

$$\mathcal{H} = \pi\,\dot{\eta}\,(x) - \mathcal{L}\,. \tag{7.6}$$

On-shell it becomes

$$\mathcal{H} = \frac{\partial}{\partial t}\overset{\neg}{\eta}(x)\,\frac{\partial}{\partial t}\eta(x)\,. \tag{7.7}$$

The commutative/anticommutative properties of the $c_\beta(\mathbf{p})$ and $c^\dagger_\beta(\mathbf{p})$ follow from considering the field energy:

$$H = \int \mathrm{d}^3 x\, \mathcal{H} \tag{7.8}$$

$$= \int \mathrm{d}^3 x \sum_{\beta,\beta'}\int\int \frac{\mathrm{d}^3 p'}{(2\pi)^3}\frac{a(p'^\mu)}{2\,E(\mathbf{p}')}\;\frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{a(p^\mu)}{2\,E(\mathbf{p})}$$
$$\times\left[c^\dagger_\beta(\mathbf{p})\,\overset{\neg\mathrm{S}}{\lambda}_\beta(\mathbf{p})(+\mathrm{i}E(\mathbf{p}))\mathrm{e}^{+\mathrm{i}p_\mu x^\mu} + c_\beta(\mathbf{p})\,\overset{\neg\mathrm{A}}{\lambda}_\beta(\mathbf{p})(-\mathrm{i}E(\mathbf{p}))\mathrm{e}^{-\mathrm{i}p_\mu x^\mu}\right]$$
$$\times\left[c_{\beta'}(\mathbf{p}')\lambda^{\mathrm{S}}_{\beta'}(\mathbf{p}')(-\mathrm{i}E(\mathbf{p}'))\mathrm{e}^{-\mathrm{i}p'_\mu x^\mu} + c^\dagger_{\beta'}(\mathbf{p}')\lambda^{\mathrm{A}}_{\beta'}(\mathbf{p}')(+\mathrm{i}E(\mathbf{p}'))\mathrm{e}^{+\mathrm{i}p'_\mu x^\mu}\right]. \tag{7.9}$$

The spatial integration gives $(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')$, resulting in

$$H = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{a^2(p^\mu)}{4}\sum_\beta \left(c^\dagger_\beta(\mathbf{p})c_\beta(\mathbf{p}) \stackrel{\neg\mathrm{S}}{\lambda_\beta}(\mathbf{p})\lambda^S_\beta(\mathbf{p}) + c_\beta(\mathbf{p})c^\dagger_\beta(\mathbf{p}) \stackrel{\neg\mathrm{A}}{\lambda_\beta}(\mathbf{p})\lambda^A_\beta(\mathbf{p})\right) . \tag{7.10}$$

Using the results (3.23) and (3.24), we get

$$H = \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{a^2(p^\mu)\,m}{2}\sum_\beta \left(c^\dagger_\beta(\mathbf{p})c_\beta(\mathbf{p}) - c_\beta(\mathbf{p})c^\dagger_\beta(\mathbf{p})\right) . \tag{7.11}$$

While we have been careful about ordering of the various operators, no specific commutators/anticommutators have been assumed. It is clear that commutative relations between $c_\beta(\mathbf{p})$ and $c^\dagger_\beta(\mathbf{p})$ will yield a field energy which vanishes for a general configuration after the usual normal ordering. For this reason the associated statistics must be fermionic:

$$\left\{c_\beta(\mathbf{p}),\ c^\dagger_{\beta'}(\mathbf{p}')\right\} = b(p^\mu)\ (2\pi)^3\, 2E(\mathbf{p})\delta^3\left(\mathbf{p}-\mathbf{p}'\right)\delta_{\beta\beta'} \tag{7.12}$$

where $b(p^\mu)\in\mathbb{R}$ will be dictated by the interpretation of H. Implementing this anticommutator on H results in

$$H = -\delta^3\left(\mathbf{0}\right)\int \mathrm{d}^3p\ \frac{a^2(p^\mu)\,m\,b(p^\mu)}{2}\sum_\beta 2E(\mathbf{p})\,\delta_{\beta\beta} + \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{a^2(p^\mu)\,m}{2}\sum_\beta 2c^\dagger_\beta(\mathbf{p})c_\beta(\mathbf{p})\,. \tag{7.13}$$

The factors of 2 in each of the $\sum_\beta$ occur because both the self-conjugate as well as the anti-self-conjugate parts of the field contribute. With this observation in mind, and in order to obtain the zero-point energy which is consistent with fermionic fields, we fix $a(p^\mu)$ and $b(p^\mu)$ by the requirement

$$\frac{a^2(p^\mu)\,m}{2} = E(\mathbf{p})\,, \tag{7.14}$$

$$\frac{a^2(p^\mu)\,m\,b(p^\mu)}{2} = \frac{1}{2}\,. \tag{7.15}$$

That is,

$$a(p^\mu) = \sqrt{\frac{2E(\mathbf{p})}{m}}\,,\qquad b(p^\mu) = \frac{1}{2E(\mathbf{p})}\,. \tag{7.16}$$

Thus, we have

$$H = -\delta^3\left(\mathbf{0}\right)\frac{1}{2}\int \mathrm{d}^3p\ \sum_\beta 2E(\mathbf{p}) + \int\frac{\mathrm{d}^3p}{(2\pi)^3}E(\mathbf{p})\sum_\beta 2c^\dagger_\beta(\mathbf{p})c_\beta(\mathbf{p})\,. \tag{7.17}$$

Since $\delta^3(\mathbf{p}) = \left[1/(2\pi)^3\right]\int\mathrm{d}^3x\exp(\mathrm{i}\mathbf{p}\cdot\mathbf{x})$, formally

$$\delta^3(\mathbf{0}) = \frac{1}{(2\pi)^3}\int\mathrm{d}^3x, \tag{7.18}$$

showing the first term in H to be

$$H_0 = -\frac{1}{(2\pi)^3}\int \mathrm{d}^3x \int \mathrm{d}^3p \, \sum_\beta 2\left(\frac{1}{2}E(\mathbf{p})\right) . \tag{7.19}$$

Since in natural units $\hbar = 1$ implies $2\pi = h$, H_0 represents an energy assignment of $-\frac{1}{2}E(\mathbf{p})$, for each helicity of the self-conjugate and anti-self-conjugate degrees of freedom (hence the factor of 2), to each unit-size phase cell $(1/h^3)\mathrm{d}^3x\mathrm{d}^3p$ in the sense of statistical mechanics. The new zero-point energy, just as in the Dirac case, comes with a minus sign, and is infinite[24]. The second term in H of equation (7.17) shows that each of the four degrees of freedom — self-conjugate: $\{-,+\},\{+,-\}$, and anti-self-conjugate: $\{-,+\},\{+,-\}$ — contributes exactly the same energy $E(\mathbf{p})$ to the field for a given momentum $\mathbf{p}$.

We thus have the result

$$\left\{c_\beta(\mathbf{p}),\ c^\dagger_{\beta'}(\mathbf{p}')\right\} = (2\pi)^3\,\delta^3\left(\mathbf{p}-\mathbf{p}'\right)\delta_{\beta\beta'} . \tag{7.20}$$

These require imposing[25]

$$\left\{c^\dagger_\beta(\mathbf{p}),\ c^\dagger_{\beta'}(\mathbf{p}')\right\} = \left\{c_\beta(\mathbf{p}),\ c_{\beta'}(\mathbf{p}')\right\} = 0 \tag{7.21}$$

to obtain correct particle interpretation. Using equation (7.14), the field operator and its Elko dual become

$$\eta(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\,m\,E(\mathbf{p})}}\sum_\beta \left[c_\beta(\mathbf{p})\lambda^{\mathrm{S}}_\beta(\mathbf{p})\mathrm{e}^{-\mathrm{i}p_\mu x^\mu} + c^\dagger_\beta(\mathbf{p})\lambda^{\mathrm{A}}_\beta(\mathbf{p})\mathrm{e}^{+\mathrm{i}p_\mu x^\mu}\right] , \tag{7.22}$$

$$\overset{\neg}{\eta}(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\,m\,E(\mathbf{p})}}\sum_\beta \left[c^\dagger_\beta(\mathbf{p})\,\overset{\neg\mathrm{S}}{\lambda_\beta}(\mathbf{p})\mathrm{e}^{+\mathrm{i}p_\mu x^\mu} + c_\beta(\mathbf{p})\,\overset{\neg\mathrm{A}}{\lambda_\beta}(\mathbf{p})\mathrm{e}^{-\mathrm{i}p_\mu x^\mu}\right] . \tag{7.23}$$

We thus establish the results with which the previous section opened.

8. Locality structure

In this section we present the three fundamental anticommutators required to state the locality structure of the theory under consideration and remark on the massless limit.

8.1 Fundamental anticommutators for the Elko quantum field

The emergent non-locality is at the second order. That is, while the field–momentum anticommutator exhibits the usual form expected of a local quantum field theory, the field–field and momentum–momentum anticommutators do not vanish.

[24] Our discussion of the new zero-point energy is similar to Zee's in [87] for the Dirac case.

[25] Actually, as we will see below, the anticommutator of $\eta(x)$ with itself turns out to be non-vanishing. Thus, it is conceivable to modify (7.21) with the idea of introducing extra terms which do not touch the locality result (8.8) below, but which cancel the terms in $\{\eta(x),\eta(x')\}$. Note that in the derivation of positivity of the Hamiltonian and in the evaluation of the propagator below only (6.3) is employed. However, this idea does not seem to work. Thus, it appears to be reasonable to keep (7.21) in order to obtain the standard fermionic harmonic oscillator algebra for $c_\beta(\mathbf{p}), c^\dagger_\beta(\mathbf{p})$.

8.1.1 Field–momentum anticommutator

We begin with the equal-time anticommutator

$$\left\{\eta(\mathbf{x},t),\, \pi(\mathbf{x}',t)\right\} = \left\{\eta(\mathbf{x},t),\, \frac{\partial}{\partial t}\,\overset{\neg}{\eta}\,(\mathbf{x}',t)\right\}. \tag{8.1}$$

The anticommutator on the right-hand side of the above equation expands to

$$\begin{aligned}\int & \frac{\mathrm{d}^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2\,m\,E(\mathbf{p})}}\int\frac{\mathrm{d}^3p'}{(2\pi)^3}\,\frac{1}{\sqrt{2\,m\,E(\mathbf{p}')}}\\ &\times\sum_{\beta\beta'}\Bigg[\Big[c_\beta(\mathbf{p})\lambda^{\mathrm{S}}_\beta(\mathbf{p})\mathrm{e}^{-\mathrm{i}p_\mu x^\mu}+c^\dagger_\beta(\mathbf{p})\lambda^{\mathrm{A}}_\beta(\mathbf{p})\mathrm{e}^{+\mathrm{i}p_\mu x^\mu}\Big]\\ &\times\Big[c^\dagger_{\beta'}(\mathbf{p}')\,\overset{\neg\mathrm{S}}{\lambda_{\beta'}}\,(\mathbf{p}')\,(+\mathrm{i}E(\mathbf{p}'))\mathrm{e}^{+\mathrm{i}p'_\mu x'^\mu}+c_{\beta'}(\mathbf{p}')\,\overset{\neg\mathrm{A}}{\lambda_{\beta'}}\,(\mathbf{p}')\,(-\mathrm{i}E(\mathbf{p}'))\mathrm{e}^{-\mathrm{i}p'_\mu x^\mu}\Big]\\ &+\Bigg[c^\dagger_{\beta'}(\mathbf{p}')\left(\overset{\neg\mathrm{S}}{\lambda_{\beta'}}\,(\mathbf{p}')\right)^\dagger(+\mathrm{i}E(\mathbf{p}'))\mathrm{e}^{+ip'_\mu x'^\mu}+c_{\beta'}(\mathbf{p}')\left(\overset{\neg\mathrm{A}}{\lambda_{\beta'}}\,(\mathbf{p}')\right)^\dagger(-\mathrm{i}E(\mathbf{p}'))\mathrm{e}^{-ip'_\mu x^\mu}\Bigg]\\ &\times\Big[c_\beta(\mathbf{p})\left(\lambda^{\mathrm{S}}_\beta(\mathbf{p})\right)^\dagger\mathrm{e}^{-\mathrm{i}p_\mu x^\mu}+c^\dagger_\beta(\mathbf{p})\left(\lambda^{\mathrm{A}}_\beta(\mathbf{p})\right)^\dagger\mathrm{e}^{+\mathrm{i}p_\mu x^\mu}\Big]\Bigg].\end{aligned} \tag{8.2}$$

Due to the anticommutators imposed by the particle interpretation, i.e., equations (6.3) and (6.4), the only terms which contribute are

$$\begin{aligned}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2\,m\,E(\mathbf{p})}}\int\frac{\mathrm{d}^3p'}{(2\pi)^3}\,\frac{1}{\sqrt{2\,m\,E(\mathbf{p}')}}\sum_{\beta\beta'}\mathrm{i}E(\mathbf{p}')&\\ \times\Big[\lambda^{\mathrm{S}}_\beta(\mathbf{p})\,\overset{\neg\mathrm{S}}{\lambda_{\beta'}}\,(\mathbf{p}')\left\{c_\beta(\mathbf{p}),c^\dagger_{\beta'}(\mathbf{p}')\right\}\mathrm{e}^{-\mathrm{i}p_\mu x^\mu+\mathrm{i}p'_\mu x'^\mu}&\\ -\lambda^{\mathrm{A}}_\beta(\mathbf{p})\,\overset{\neg\mathrm{A}}{\lambda_{\beta'}}\,(\mathbf{p}')\left\{c^\dagger_\beta(\mathbf{p}),c_{\beta'}(\mathbf{p}')\right\}\mathrm{e}^{+\mathrm{i}p_\mu x^\mu-\mathrm{i}p'_\mu x'^\mu}\Big]&.\end{aligned} \tag{8.3}$$

Here, we have used the fact that

$$\left(\overset{\neg\mathrm{S/A}}{\lambda_\beta}\,(\mathbf{p})\right)^\dagger\left(\lambda^{\mathrm{S/A}}_\beta(\mathbf{p})\right)^\dagger=\left(\lambda^{\mathrm{S/A}}_\beta(\mathbf{p})\,\overset{\neg\mathrm{S/A}}{\lambda_\beta}\,(\mathbf{p})\right)^\dagger=\lambda^{\mathrm{S/A}}_\beta(\mathbf{p})\,\overset{\neg\mathrm{S/A}}{\lambda_\beta}\,(\mathbf{p}). \tag{8.4}$$

Using (6.3) and (6.4) and performing the $\mathbf{p}'$ integration reduces (8.3) to

$$\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\frac{\mathrm{i}}{2\,m}\sum_\beta\Big[\lambda^{\mathrm{S}}_\beta(\mathbf{p})\,\overset{\neg\mathrm{S}}{\lambda_\beta}\,(\mathbf{p})\mathrm{e}^{-\mathrm{i}p_\mu x^\mu+\mathrm{i}p_\mu x'^\mu}-\lambda^{\mathrm{A}}_\beta(\mathbf{p})\,\overset{\neg\mathrm{A}}{\lambda_\beta}\,(\mathbf{p})\mathrm{e}^{+\mathrm{i}p_\mu x^\mu-\mathrm{i}p_\mu x'^\mu}\Big]. \tag{8.5}$$

Now first note that we are calculating an equal-time anticommutator. So, set $t'=t$. Next, in the second terms change integration variable from $\mathbf{p}$ to $-\mathbf{p}$. This makes (8.5) to be

$$\frac{\mathrm{i}}{2\,m}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\sum_\beta\left[\lambda^{\mathrm{S}}_\beta(\mathbf{p})\,\overset{\neg\mathrm{S}}{\lambda_\beta}\,(\mathbf{p})-\lambda^{\mathrm{A}}_\beta(-\mathbf{p})\,\overset{\neg\mathrm{A}}{\lambda_\beta}\,(-\mathbf{p})\right]. \tag{8.6}$$

The indicated spin sum equals $2\,m\,(\mathbb{I}+\mathcal{G}(\phi))$. This simplifies (8.6) to

$$\mathrm{i}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}+\mathrm{i}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\mathcal{G}(\phi). \tag{8.7}$$

The second integral vanishes because in the absence of a preferred direction we are free to choose $\mathbf{x} - \mathbf{x}'$ to be aligned with the z-axis, and thus the inner product with $\mathbf{p}$ becomes independent of ϕ, while an integration of $\mathcal{G}(\phi)$ over one period vanishes. Therefore, we obtain

$$\left\{\eta(\mathbf{x},t),\, \pi(\mathbf{x}',t)\right\} = \mathrm{i}\,\delta^3\left(\mathbf{x}-\mathbf{x}'\right) . \tag{8.8}$$

At this point it is emphasized that the 'standard result' (8.8) ceases to be valid in the presence of a preferred direction, in general.

8.1.2 Field–field, and momentum–momentum, anticommutators

By inspection of the above calculation one readily obtains

$$\begin{aligned}&\left\{\eta(\mathbf{x},t),\, \overset{\neg}{\eta}\,(\mathbf{x}',t)\right\}\\ &\qquad = \frac{1}{2\,m}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\,\frac{1}{E(\mathbf{p})}\,\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\sum_\beta \left[\lambda^{\mathrm{S}}_\beta(\mathbf{p})\,\overset{\neg}{\lambda}{}^{\mathrm{S}}_\beta(\mathbf{p}) + \lambda^{\mathrm{A}}_\beta(-\mathbf{p})\,\overset{\neg}{\lambda}{}^{\mathrm{A}}_\beta(-\mathbf{p})\right] .\end{aligned} \tag{8.9}$$

The spin sum on the right-hand side of the above equation vanishes identically, giving

$$\left\{\eta(\mathbf{x},t),\, \overset{\neg}{\eta}\,(\mathbf{x}',t)\right\} = 0\,. \tag{8.10}$$

However, as we shall now show,

$$\left\{\eta(\mathbf{x},t), \eta(\mathbf{x}',t)\right\} \neq 0\,. \tag{8.11}$$

It turns out that not only is the result non-vanishing, but it depends on anticommutators of creation and annihilation operators, and also on commutators. We, therefore, evaluate its vacuum expectation value:

$$\begin{aligned}\Big\langle\quad\Big|\left\{\eta(\mathbf{x},t), \eta(\mathbf{x}',t)\right\}\Big|\quad\Big\rangle &= \int\frac{\mathrm{d}^3 p}{(2\pi)^3}\,\frac{1}{\sqrt{2\,m\,E(\mathbf{p})}}\int\frac{\mathrm{d}^3 p'}{(2\pi)^3}\,\frac{1}{\sqrt{2\,m\,E(\mathbf{p}')}}\\ &\quad\times\sum_{\beta\beta'}\Big[\left\{c_\beta(\mathbf{p}),\, c^\dagger_{\beta'}(\mathbf{p}')\right\}\lambda^{\mathrm{S}}_\beta(\mathbf{p})\left(\lambda^{\mathrm{A}}_{\beta'}(\mathbf{p}')\right)^{\mathrm{T}}\mathrm{e}^{-\mathrm{i}p_\mu x^\mu + \mathrm{i}p'_\mu x'^\mu}\\ &\quad+\left\{c_{\beta'}(\mathbf{p}'),\, c^\dagger_\beta(\mathbf{p})\right\}\lambda^{S}_{\beta'}(\mathbf{p}')\left(\lambda^{\mathrm{A}}_\beta(\mathbf{p})\right)^{\mathrm{T}}\mathrm{e}^{-\mathrm{i}p'_\mu x'^\mu + \mathrm{i}p_\mu x^\mu}\Big] ,\end{aligned} \tag{8.12}$$

which exhibits only anticommutators and thus yields

$$\begin{aligned}&\Big\langle\quad\Big|\left\{\eta(\mathbf{x},t), \eta(\mathbf{x}',t)\right\}\Big|\quad\Big\rangle\\ &= \frac{1}{2\,m}\int\frac{\mathrm{d}^3 p}{(2\pi)^3}\,\frac{1}{E(\mathbf{p})}\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\sum_\beta\left[\lambda^{\mathrm{S}}_\beta(\mathbf{p})\left(\lambda^{\mathrm{A}}_\beta(\mathbf{p})\right)^{\mathrm{T}} + \lambda^{\mathrm{S}}_\beta(-\mathbf{p})\left(\lambda^{\mathrm{A}}_\beta(-\mathbf{p})\right)^{T}\right] .\end{aligned} \tag{8.13}$$

Inserting (B.20) and performing the ϕ integration yields the expression

$$\frac{1}{4\,m\,\pi^2}\int_0^\infty \mathrm{d}p\,\frac{p^2}{\sqrt{p^2+m^2}}\int_0^\pi \mathrm{d}\theta\,\sin(\theta)\,\mathrm{e}^{\mathrm{i}pr\cos(\theta)}\left(p\sin\theta\gamma^0\gamma^1 + \sqrt{p^2+m^2}\gamma^2\gamma^5\right) . \tag{8.14}$$

For the θ integration the following two integrals are needed:

$$\int_0^\pi \mathrm{d}\theta \, \sin^2(\theta) \, e^{\mathrm{i}\, p\, r\, \cos(\theta)} = \frac{\pi J_1(p\, r)}{p\, r} \,, \tag{8.15}$$

$$\int_0^\pi \mathrm{d}\theta \, \sin(\theta) \, \mathrm{e}^{\mathrm{i}\, p\, r\, \cos(\theta)} = \frac{2\, \sin(p\, r)}{p\, r} \,, \tag{8.16}$$

where J_1 is the Bessel function of the first kind. This implies

$$\Big\langle \; \Big| \{\eta(\mathbf{x},t), \eta(\mathbf{x}',t)\} \Big| \; \Big\rangle = \frac{1}{4\, m\, \pi\, r^3}$$
$$\times \left[\int_0^\infty \mathrm{d}(p\, r) \left\{ \frac{(p\, r)^2 \, J_1(p\, r)}{\sqrt{(p\, r)^2 + (m\, r)^2}} \, \gamma^0\gamma^1 + \frac{2}{\pi} p\, r \, \sin(p\, r) \, \gamma^2\gamma^5 \right\} \right] . \tag{8.17}$$

For the final p integration the following two integrals have to be evaluated:

$$\int_0^\infty \frac{x^2 J_1(x)}{\sqrt{x^2 + (mr)^2}} \mathrm{d}x = e^{-mr}(1 + mr) \,, \tag{8.18}$$

and

$$\int_0^\infty p \sin{(p\, r)} \mathrm{d}p = \pi \delta(r^2) \,, \tag{8.19}$$

where the last result is only true in a distributional sense, cf appendix B.5.

Therefore, the final result is established:

$$\Big\langle \; \big| \{\eta(\mathbf{x},t), \eta(\mathbf{x}',t)\} \big| \; \Big\rangle = \frac{1 + mr}{4\, m\, \pi r^3} e^{-mr} \gamma^0\gamma^1 + \frac{1}{2\, m\, \pi r^2} \delta(r^2) \gamma^2\gamma^5 \,. \tag{8.20}$$

The first term has a Yukawa-like behavior while the second one is localized. Note that $\gamma^2\gamma^5$ actually is related to Wigner's time reversal operator:

$$\frac{1}{\mathrm{i}} \gamma^2\gamma^5 = \begin{pmatrix} \mathbb{O} & \Theta \\ \Theta & \mathbb{O} \end{pmatrix} \,, \tag{8.21}$$

where Θ is given in equation (2.12) — see [88]. In order to interpret the result it is useful to consider non-locality integrated over all separations $\mathbf{x} - \mathbf{x}'$. The angular part just gives the usual $4\pi r^2$ in the measure. The relevant radial integrals are

$$\int_0^\infty \mathrm{d}r \; e^{-mr} = \frac{1}{m} \,, \tag{8.22}$$

$$\int_{r_0}^\infty \mathrm{d}r \; \frac{e^{-mr}}{r} = -\gamma - \ln{(mr_0)} + \mathcal{O}(mr_0) \,, \tag{8.23}$$

$$\int_0^\infty \mathrm{d}r \; \delta(r^2) = \frac{1}{2} \int_{-\infty}^\infty \mathrm{d}r \; \delta(r^2) = \frac{1}{4} \int_{-\infty}^\infty \frac{\delta(r)}{|r|} \, \mathrm{d}r \,, \tag{8.24}$$

where γ is the Euler–Mascheroni constant. Obviously, the main (singular) contribution comes from the coincidence limit, but the exponential tail does contribute to the finite part. This singular behavior is to be contrasted with the regularity of ordinary locality

(8.8). Note that we had to regularize one of the integrals. Thus, in order to obtain a convergent result which is independent from the cutoff r_0, we consider the quantity

$$\frac{\mathrm{d}}{\mathrm{d}m}\left[m\int_{\mathbf{x}-\mathbf{x}'}\Big\langle\;\Big|\left\{\eta(\mathbf{x},t),\eta(\mathbf{x}',t)\right\}\Big|\;\Big\rangle\right]=\frac{1}{m}\gamma^1\gamma^0\,. \tag{8.25}$$

It is non-trivial that both the divergent contribution and the regulator, r_0, drop out after multiplication with m and taking the derivative with respect to m. The quantity (8.25) may be used to estimate the sensitivity of non-locality to mass. In the large-m limit, non-locality becomes negligible.

The evaluation of

$$\Big\langle\;\Big|\left\{\pi(\mathbf{x},t),\,\pi(\mathbf{x}',t)\right\}\Big|\;\Big\rangle=\Big\langle\;\Big|\left\{\frac{\partial}{\partial t}\stackrel{\neg}{\eta}(\mathbf{x},t),\,\frac{\partial}{\partial t}\stackrel{\neg}{\eta}(\mathbf{x}',t)\right\}\Big|\;\Big\rangle \tag{8.26}$$

can be performed in full analogy to the previous case. Again it is found that the vacuum expectation value is non-trivial, exhibiting non-locality and containing a distributional part.

8.2 Massless limit and non-locality

It is clear from the above discussion that the massless limit of the quantum field theory based on *Elko* is singular. Specifically, it can be seen from the anticommutators studied in section 8.1.2. Therefore, the massless-limit decoupling of the the right and left transforming components of *Elko* at the representation space level may be misleading to some extent. This is due to the fact that the *Elko* field $\eta(\mathbf{x},\,t)$, and the associated momentum $\pi(\mathbf{x},\,t)$, together carry additional information to that contained in the underlying representation space alone.

It is also our duty to bring to our reader's attention that according to a pioneering work of Weinberg [89] all $(j,0)\oplus(0,j)$ quantum fields, independent of spin, enjoy a smooth non-singular massless limit. That our result disagrees with this expected wisdom is quite simple: Weinberg's analysis explicitly assumes locality[26].

8.3 Signatures of Elko non-locality in the physical amplitudes and cross-sections

It is worthwhile recalling that the field anticommutator between $\eta(x)$ and the conjugate momentum $\pi(x)$ as obtained in equation (8.8) displays the behavior expected from a *local* quantum field theory. It is only in the anticommutators of $\eta(x)$ (or $\pi(x)$) with itself that non-locality emerges. Technically, non-locality is a direct consequence of the non-triviality of *Elko* spin sums.

The question now arises if the *Elko* non-locality carries its signatures in the physical amplitudes and cross-sections. That the answer to the question is non-trivial is already hinted at by the fact that *Elko* spin sums which underlie non-locality end up modifying the fermionic propagator (of *Elko*). In the absence of a preferred direction, this modification to the amplitude for propagation from one spacetime point to another exhibits itself not as

[26] Section 8.2 was added to the manuscript as an answer to a question by Yong Liu [90].

a non-local propagation but as a change in the mass dimensionality of the field. Yet, the non-local anticommutators suggest that there may be non-vanishing physical correlations for *Elko* events carrying spacelike separation. In the context of the physics of very early universe this, if born out by relevant S-matrix framework, may have significant bearing on the horizon problem. The latter asks for causal thermal contact between spacelike separated regions of the very early universe.

To address this possibility we here undertake a preliminary investigation of the effect of interactions on physical amplitudes and cross-sections. We find that non-locality affects these, and hence we establish that it is endowed with observable physical consequences in a non-trivial manner. In particular, it will be shown that non-locality plays a decisive role for higher-order correlation functions.

To this end we split the total Hamiltonian H into free and interaction parts, $H = H_0 + H_{\rm int}$, where H_0 is given by (7.11). For $H_{\rm int}$ one may take for example the last term in (9.2) or (9.3) (integrated over space and with a relative minus sign, because we need the Hamiltonian rather than the Lagrangian density). One may now employ the free Hamiltonian to construct the free Heisenberg fields,

$$\eta_0^H(t,\mathbf{x}) = \mathrm{e}^{\mathrm{i}H_0(t-t_0)/2}\eta(t_0,\mathbf{x})\mathrm{e}^{-\mathrm{i}H_0(t-t_0)/2}\,, \tag{8.27}$$

where $\eta(t_0,\mathbf{x})$ is given by equation (7.22) with t replaced by t_0. The unusual factors of $1/2$ in the exponent should be noted; they are needed, as will be shown below. From equation (7.11) it is evident that $H_0^\dagger = H_0$; unitarity holds also for the interaction Hamiltonian in (9.2) or (9.3) because from equation (3.23) and equation (3.24) it may be deduced that

$$\left(\stackrel{\neg}{\eta}(x)\eta(x)\right)^\dagger = \stackrel{\neg}{\eta}(x)\eta(x)\,. \tag{8.28}$$

In order to verify that η_0^H coincides with the free-field expression (7.22) the relations ($n\in\mathbb{N}$)

$$(H_0)^n c_\alpha(\mathbf{p}) = c_\alpha(\mathbf{p})\left(H_0 - 2E(\mathbf{p})\right)^n\,, \qquad (H_0)^n c_\alpha^\dagger(\mathbf{p}) = c_\alpha^\dagger(\mathbf{p})\left(H_0 + 2E(\mathbf{p})\right)^n \tag{8.29}$$

are very helpful. Again something unusual happens with factors: for an ordinary Klein–Gordon field one obtains instead $H_0 \pm E(\mathbf{p})$ in the expressions on the right-hand side. These factors of 2 are a consequence of the presence of two Elko-modes: self-dual and anti-self-dual ones[27].

The full Heisenberg field may now be defined as

$$\eta^H(t,\mathbf{x}) = U^\dagger(t,t_0)\eta_0^H(t,\mathbf{x})U(t,t_0) \tag{8.30}$$

with the time-evolution operator

$$U(t,t_0) = \mathrm{e}^{\mathrm{i}H_0(t-t_0)/2}\mathrm{e}^{-\mathrm{i}H(t-t_0)/2}\,. \tag{8.31}$$

[27]Of course it is possible to change the definition of a in (7.14) such that the factors $1/2$ and 2 disappear. But this will change the normalization of η and π and thus the canonical anticommutator (8.8) would acquire an unwanted factor of $1/2$.

Standard methods lead to the well known result (again with a somewhat unusual factor of 1/2):

$$U(t,t') = \mathcal{T}\,\exp\left(-\mathrm{i}\int_{t'}^{t} d\tilde{t}\; H^{H}_{\mathrm{int}}(\tilde{t})/2\right), \tag{8.32}$$

where $H^{H}_{\mathrm{int}}(\tilde{t})$ is the interaction Hamiltonian written as Heisenberg operator with respect to the free fields ('interaction picture') and $\mathcal{T}$ denotes time ordering. The time-evolution operator fulfills the Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}U(t,t') = \frac{H^{H}_{\mathrm{int}}(t)}{2}U(t,t')\,. \tag{8.33}$$

With the initial condition $U(t,t) = 1$, its solution is given by

$$U(t,t') = e^{\mathrm{i}H_0(t-t_0)/2}\mathrm{e}^{-\mathrm{i}H(t-t')/2}\mathrm{e}^{-\mathrm{i}H_0(t'-t_0)/2}\,. \tag{8.34}$$

Note that $U(t,t_0)U(t_0,t') = U(t,t')$ and $U(t,t_0)(U(t',t_0))^{\dagger} = U(t,t')$, as expected.

Thus, as in standard perturbative quantum field theory one may express correlation functions (with respect to the ground state of the interacting theory) between interacting fields in terms of (a Taylor series of) correlation functions (with respect to the ground state of the free theory) of free Heisenberg fields. Therefore it is sufficient to consider correlators of the form

$$\Big\langle\ \Big|\mathcal{T}\left\{\overset{\neg}{\eta}{}^{H}_{0}(x_1)\eta^{H}_{0}(x_1)\,\overset{\neg}{\eta}{}^{H}_{0}(x_2)\eta^{H}_{0}(x_2)\cdots\overset{\neg}{\eta}{}^{H}_{0}(x_n)\eta^{H}_{0}(x_n)\right\}\Big|\ \Big\rangle\,. \tag{8.35}$$

The next step is, by analogy to Wick's theorem, to study the relation between the time-ordered product of correlators and the normal-ordered product. We will not attempt to do this in full generality but rather restrict ourselves to the bilinear case and address below essential features of the quartic case. The notation

$$\eta^{+} := \mathcal{I}_\beta c_\beta \lambda^{\mathrm{S}}_\beta(\mathbf{p})\mathrm{e}^{-\mathrm{i}px}\,, \qquad \eta^{-} := \mathcal{I}_\beta c^{\dagger}_\beta \lambda^{\mathrm{A}}_\beta(\mathbf{p})\mathrm{e}^{\mathrm{i}px}\,, \tag{8.36}$$

$$\overset{\neg}{\eta}{}^{+} := \mathcal{I}_\beta c_\beta \overset{\neg}{\lambda}{}^{\mathrm{A}}_\beta(\mathbf{p})\mathrm{e}^{-\mathrm{i}px}\,, \qquad \overset{\neg}{\eta}{}^{-} := \mathcal{I}_\beta c^{\dagger}_\beta \overset{\neg}{\lambda}{}^{\mathrm{S}}_\beta(\mathbf{p})\mathrm{e}^{\mathrm{i}px}\,, \tag{8.37}$$

with the abbreviation

$$\mathcal{I}_\beta := \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2mE(\mathbf{p})}}\sum_\beta\,, \tag{8.38}$$

allows the decomposition

$$\eta = \eta^{+} + \eta^{-}\,, \qquad \overset{\neg}{\eta} = \overset{\neg}{\eta}{}^{+} + \overset{\neg}{\eta}{}^{-}\,. \tag{8.39}$$

For $x^0 > y^0$ one obtains

$$\mathcal{T}\left(\eta(x)\,\overset{\neg}{\eta}(y)\right) = \mathcal{N}\left(\eta(x)\,\overset{\neg}{\eta}(y)\right) + \{\eta^{+}(x), \overset{\neg}{\eta}{}^{-}(y)\}\,, \tag{8.40}$$

where $\mathcal{N}$ denotes (fermionic) normal ordering, i.e., all c_β operators are on the right-hand side and all $c_\beta^\dagger$ operators on the left-hand side in each expression. By virtue of (5.55), the anticommutator evaluates to[28]

$$\{\eta^+(x), \stackrel{\neg}{\eta}^{-}(y)\} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\mathbb{I}+\mathcal{G}}{2E(\mathbf{p})} \mathrm{e}^{-\mathrm{i}p(x-y)} \,. \tag{8.41}$$

For $x^0 < y^0$ analogous steps yield

$$\mathcal{T}\Big(\eta(x) \stackrel{\neg}{\eta}(y)\Big) = \mathcal{N}\Big(\eta(x) \stackrel{\neg}{\eta}(y)\Big) + \{\eta^-(x), \stackrel{\neg}{\eta}^{+}(y)\} \,. \tag{8.42}$$

The spin sum (5.56) yields

$$\{\eta^-(x), \stackrel{\neg}{\eta}^{+}(y)\} = -\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\mathbb{I}-\mathcal{G}}{2E(\mathbf{p})} \mathrm{e}^{\mathrm{i}p(x-y)} \,. \tag{8.43}$$

Comparison with (6.14) establishes a standard result: time-ordering decomposes into normal-ordering and contraction, where the latter procedure leads to i times the Elko-propagator $\mathcal{S}_{\mathrm{FD}}^{\mathrm{Elko}}(x,y)$, which will be recalled for convenience:

$$\mathcal{S}_{\mathrm{FD}}^{\mathrm{Elko}}(x,y) = \lim_{\epsilon \to 0^+} \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\, \mathrm{e}^{-\mathrm{i}p_\mu(x^\mu - y^\mu)} \left[\frac{(\mathbb{I}+\mathcal{G}(\phi))}{p_\mu p^\mu - m^2 + \mathrm{i}\epsilon}\right] . \tag{8.44}$$

It equals the Klein–Gordon propagator only in the absence of a preferred direction. Thus, in the presence of some non-trivial background (which may be provided also by Elko itself in higher-order perturbation theory) the terms proportional to $\mathcal{G}$ no longer will be negligible.

In order to unravel the appearance of non-locality one has to study higher-order correlation functions. Therefore we now address the quartic expression

$$\mathcal{T}\Big(\eta(x) \stackrel{\neg}{\eta}(y) \eta(z) \stackrel{\neg}{\eta}(w)\Big) = \mathcal{N}\Big(\eta(x) \stackrel{\neg}{\eta}(y) \eta(z) \stackrel{\neg}{\eta}(w)\Big) + \text{contractions} \,. \tag{8.45}$$

The decomposition into $\pm$ components yields 16 terms which we denote by $[\pm\pm\pm\pm]$, 5 of which are already normal-ordered ($[++++], [-+++], [--++], [---+], [----]$). In two terms a single contraction of the type (8.41) (or (8.43)) is sufficient ($[+-++]$, $[--+-]$) and in two terms two contractions of that type appear ($[+-+-]$, $[-+-+]$). But in the remaining seven terms an additional feature arises which is unprecedented, namely, is absent for Dirac fermions. We will discuss in full detail one of these seven terms, $[++-+]$, in order to make this peculiarity explicit and to establish the connection with non-locality:

$$\begin{aligned}
&\eta^+(x) \stackrel{\neg}{\eta}^{+}(y) \;\; \eta^-(z) \stackrel{\neg}{\eta}^{+}(w) = -\eta^+(x)\eta^-(z) \stackrel{\neg}{\eta}^{+}(y) \stackrel{\neg}{\eta}^{+}(w) + \eta^+(x)\{\eta^-(z), \stackrel{\neg}{\eta}^{+}(y)\} \stackrel{\neg}{\eta}^{+}(w) \\
&= \eta^-(z)\eta^+(x) \stackrel{\neg}{\eta}^{+}(y) \stackrel{\neg}{\eta}^{+}(w) - \{\eta^+(x), \eta^-(z)\} \stackrel{\neg}{\eta}^{+}(y) \stackrel{\neg}{\eta}^{+}(w) + \eta^+(x)(\text{prop. part}) \stackrel{\neg}{\eta}^{+}(w) \\
&= \mathcal{N}\Big(\eta^+(x) \stackrel{\neg}{\eta}^{+}(y) \eta^-(z) \stackrel{\neg}{\eta}^{+}(w)\Big) - \{\eta^+(x), \eta^-(z)\} \stackrel{\neg}{\eta}^{+}(y) \stackrel{\neg}{\eta}^{+}(w) + \text{ordinary contraction.}
\end{aligned}$$

[28]As usual all anticommutators are to be understood as being matrix-valued in spinor-space, rather than scalar-valued. That is why instead of the orthonormality relations (3.23), (3.24) the spin sums (5.55), (5.56) appear subsequently.

The crucial point here is that the anticommutator in the middle term in the last line is *non-vanishing*, in stark contrast to Dirac fermions. A simple calculation yields

$$\{\eta^{+}(x), \eta^{-}(z)\} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2mE(\mathbf{p})}} \sum_{\beta} \lambda^{\mathrm{S}}_{\beta}(\mathbf{p}) \left(\lambda^{\mathrm{A}}_{\beta}(\mathbf{p})\right)^{T} \mathrm{e}^{-\mathrm{i}p(x-z)}\,, \tag{8.46}$$

which may be evaluated following (B.20). Incidentally, this is the *same* spin sum, the non-vanishing of which had been responsible for the emergence of non-locality in (8.13).

Thus, Wick's theorem in abbreviated form reads

$$\mathcal{T}\left(\eta(x^1) \stackrel{\neg}{\eta}(x^1) \ldots \eta(x^n) \stackrel{\neg}{\eta}(x^n)\right) = \mathcal{N}\left(\eta(x^1) \stackrel{\neg}{\eta}(x^1) \ldots \eta(x^n) \stackrel{\neg}{\eta}(x^n)\right) + \text{contractions}\,, \tag{8.47}$$

where 'contractions' decompose into 'ordinary contractions' between η and $\stackrel{\neg}{\eta}$, yielding the propagator $\mathrm{i}\mathcal{S}^{\mathrm{Elko}}_{\mathrm{FD}}$, and into 'non-local contractions' between η and itself or between $\stackrel{\neg}{\eta}$ and itself. That such contractions are non-vanishing is a novel feature of Elko as compared to the Dirac case, and directly linked to the non-locality anticommutator (8.13).

Equipped with the tools of Wick's theorem (which may be derived in a standard fashion by complete induction) one may now derive the Feynman rules for Elko; no unusual features are found in this way, besides the remarkable behavior encoded in the $\mathcal{G}$-part of the Elko-propagator and the non-triviality of contractions of Elko with itself.

With these results the primary task of this subsection has been established, i.e., we have shown that for an interacting theory the physical amplitudes and cross-sections carry non-trivial signatures of non-locality. To examine the quantitative impact on the cosmological horizon problem now requires calculation of correlations between *Elko* scattering amplitudes for spacelike separated events. Such a calculation is far from trivial, but the analysis of this section makes it, in principle, well defined[29].

9. Identification of Elko with dark matter

Having established the kinematic structure of an *Elko*-based framework via the introduction of $\eta(x)$ and having studied its various properties, our first task is simply to name the particles the field $\eta(x)$ describes. We suggest the symbols ς and $\stackrel{\neg}{\varsigma}$ for these particles — with obvious symbolic distinction from the Dirac framework.

The next question which arises is what interactions can ς and $\stackrel{\neg}{\varsigma}$ carry with the Standard Model fields. Towards the goal of answering this question we note that one of the reasons for the success of the Standard Model stems from the fact that only a very limited number of terms appear in the action. Simple arguments of power counting renormalizability prohibit for example 4-Fermi interactions or non-polynomial potentials for the Higgs. While these non-renormalizable terms are not strictly forbidden, in the low-energy regime that we are able to explore experimentally they play essentially no role because they are suppressed by factors $(k/M_{\mathrm{U}})^d$ for $k \ll M_{\mathrm{U}}$ (k characterizes the low-energy scale), where $4+d$ is the

[29]This section was added on suggestion from a *JCAP* referee.

mass dimension of the interaction, and $M_{\rm U}$ is the GUT or the Planck scale (cf e.g. section 12.3 in [8]).

Having addressed the relevance of non-local contributions for *higher-order* perturbation theory in the previous section, and to proceed further, we explicitly state an observation and the *working assumption* we must make

> Non-locality of *Elko* appears to prohibit a naive application of power-counting arguments. Nevertheless, we take them as good starting point in the hope that it could well be that non-locality is a higher-order effect because, after all, the equal-time anticommutator between $\eta(x)$ and the associated canonical momentum $\pi(x)$ is local. So, non-locality manifests itself when at least two *Elko* fields or two momenta appear together in the same expectation value.

Therefore, the interactions of ς and $\stackrel{\neg}{\varsigma}$ with the Standard Model fields are entirely governed by mass dimension one of $\eta(x)$ and power-counting arguments. As a result the following additional structure for the Standard Model Lagrangian density comes to exist:

$$\mathcal{L}^{\rm new}(x) = \mathcal{L}^{\rm Elko}(x) + \mathcal{L}^{\rm int}(x)\,, \tag{9.1}$$

$$\mathcal{L}^{\rm Elko}(x) = \partial^\mu \stackrel{\neg}{\eta}(x)\partial_\mu \eta(x) - m^2 \stackrel{\neg}{\eta}(x)\eta(x) + \alpha_{\rm E}\left[\stackrel{\neg}{\eta}(x)\eta(x)\right]^2\,, \tag{9.2}$$

$$\mathcal{L}^{\rm int}_{\phi\eta}(x) = \lambda_{\rm E}\,\phi^\dagger(x)\phi(x)\,\stackrel{\neg}{\eta}(x)\eta(x)\,, \tag{9.3}$$

where $\phi(x)$ is the Higgs doublet, m the Elko mass, and $\lambda_{\rm E}, \alpha_{\rm E}$ are *dimensionless* coupling constants. Obviously, if more than one *Elko* field is present mixings between them are possible. However, as there seems to be no way to distinguish different *Elko* particles — as opposed to the Standard Model fermions which are distinguished by their charges — it is sensible to introduce only one *Elko* field. Of course, if more than one scalar field should be present in Nature additional interactions of the form (9.3) are possible. The fact that (9.1) contains no interactions with gauge fields or other Standard Model (SM) fermions explains why *Elko* has not been detected yet. However, since it does interact with the Higgs there is a possibility that it might be discovered at LHC. Clearly, a thorough analysis of this issue would be desirable.

While ς and $\stackrel{\neg}{\varsigma}$ may carry a coupling to an Abelian gauge field with associated field strength $F_{\mu\nu}(x)$, for example of the form

$$\mathcal{L}^{\rm int}_{\eta F}(x) = \epsilon_{\rm E} \stackrel{\neg}{\eta}(x)\,[\gamma^\mu, \gamma^\nu]\,\eta(x)\,F_{\mu\nu}(x)\,, \tag{9.4}$$

the coupling constant has to be very small. Such terms affect photon propagation because they lead to an effective-mass term for the photon and the latter has been severely constrained [91, 92]. The indicated smallness is not unexpected as $\eta(x)$ is neutral with respect to local $U(1)$ gauge transformations. Thus, the dominant interaction between *Elko* and particles of the Standard Model is expected to be via (9.3)[30].

[30]We are grateful to Dima Vassilevich for raising a question in this regard.

It is also worth noting that the SM-counterpart of the quartic self-interaction contained in (9.2) is suppressed as $(k/M_{\mathrm{U}})^2$ in sharp contrast to *Elko*. This *Elko* self-interaction could be of significant physical consequence, as we shall remark below.

We end this section with a conclusion that dawns on us unexpectedly: the fact that ς and $\stackrel{\neg}{\varsigma}$ almost do not interact with the matter content of the Standard Model makes them prime dark matter candidates.

10. Constraining the Elko mass and the relevant cross-section

In the next two subsections we strive to constrain the mass and the relevant cross-section of these candidate particles. While the first of these two subsections involves a conventional exercise on relic density, the second subsection studies the gravitational collapse of a primordial Elko cloud. It is encouraging that the latter calculation finds an element of consistency with the relic density analysis. The collapse, with eventual rebound, should be considered as a process that precedes virialization of the dark matter cloud. We thus consider both of these approaches complementary to an extent. The collapse analysis also predicts an explosive event in the early history of each galaxy containing dark matter. It should be considered a candidate for cosmic gamma-ray bursts [93].

10.1 Relic density of Elko: constraining the relevant cross-section, mass, and a comparison with WIMP

In a series of publication the case for a MeV-range dark matter particle has been gaining strength [94–98]. At the same time, as will be seen in section 10.2, same indication arises from an entirely different consideration. For these reasons we concentrate on the MeV-range *Elko* mass.

In its essentials the analysis that follows is an adaptation of the one presented by Bergström and Goobar [99], Dodelson [100], and Kolb and Turner [101] for weakly interacting massive particles (WIMPs). Since a comparison with a generic WIMP scenario (of which the supersymmetry-implied candidates are a subset) may be useful, we shall present the analysis in such a manner as to make this apparent[31].

A generic WIMP scenario for dark matter considers two heavy particles X which annihilate into two light particles. The light particles are considered to be tightly coupled to the cosmic plasma, so that their number density in the cosmic comoving frame equals n^{EQ}, the equilibrium value. This leaves one unknown density, i.e., that associated with the WIMP dark matter particle. Its evolution is determined by the Boltzmann equation. For *Elko* the thermodynamic situation is as follows. The thermal contact of the cosmic plasma made of the Standard Model particles with *Elko* is provided by the Higgs: $\varsigma + \stackrel{\neg}{\varsigma} \rightleftharpoons \phi_H + \phi_H$[32]. The coupling constant λ_{E} of equation (9.3) then determines the *Elko* annihilation rate through the thermally averaged product of the cross-section $\sigma_{\varsigma\stackrel{\neg}{\varsigma}\to\phi_H\phi_H}$ and the Møller

[31]For convenience, we here use units where c, $\hbar$, and Boltzmann constant k_B are set to unity.

[32]Here, ϕ_H represents a Higgs particle and shall be assumed to have a mass of roughly 150 GeV. We do not use the symbol H for Higgs particle as we wish to reserve that symbol for the Hubble rate.

velocity (which in the cosmic comoving frame can be identified with the relative velocity of the two annihilating *Elko*): $\langle \sigma_{\varsigma\,\stackrel{\neg}{\varsigma}\to\phi_H\phi_H} |\mathbf{v}_{M\o l}|\rangle$.

In the considered scenario, $m_\varsigma \ll m_{\phi_H}$, whereas for the WIMP scenario $m_X \sim m_{\phi_H}$. The quartic self-interaction of *Elko* given in equation (9.2) does *not* change the comoving density of *Elko*, n_ς, at the tree level. Instead, in conjunction with gravitational interactions a non-zero α_{E} of equation (9.2) will contribute to fluctuations in n_ς which seed the large-scale structure formation in the Cosmos.

With these remarks in mind, and under the assumption that Fermi statistics can be ignored for a zeroth-order understanding of the relic abundance of ς, the *Elko* number density n_ς is governed by[33]

$$a^{-3}\frac{\mathrm{d}\left(n_\varsigma a^3\right)}{\mathrm{d}t} = \left\langle \sigma_{\varsigma\,\stackrel{\neg}{\varsigma}\to\phi_H\phi_H} |\mathbf{v}_{M\o l}|\right\rangle \left\{\left(n_\varsigma^{\mathrm{EQ}}\right)^2 - n_\varsigma^2\right\}. \tag{10.1}$$

Here, a is the cosmic scale factor. Since CPT is respected by the *Elko* framework the n_ς divides equally among ς and $\stackrel{\neg}{\varsigma}$.[34] The symbol n_ς is used as an abbreviation for $n_{\varsigma\stackrel{\neg}{\varsigma}}$, the combined comoving number density of ς and $\stackrel{\neg}{\varsigma}$.

Before $T \approx m_{\phi_H}$, Standard Model particles and the *Elko* slosh back and forth thermally. Metaphorically speaking, Higgs serves as a two–headed fountain: on its one end it sucks and sprays the *Elko* into the cosmic plasma of the early universe and on the other it does the same for the Standard Model particles. Once the temperature falls below the Higgs mass, the high-energy tails of the *Elko* thermal distribution continues this process for a while. For a representative $m_\varsigma = 20$ MeV for ς mass, and with an example value of $\left\langle \sigma_{\varsigma\,\stackrel{\neg}{\varsigma}\to\phi_H\phi_H} |\mathbf{v}_{M\o l}|\right\rangle = 1$ pb, freeze out occurs at a temperature of about 2.1 MeV (see below). The viability of *Elko* as a serious dark matter candidate is established by showing that, for this representative example, the *Elko* energy density ρ_ς today is of the same order as the critical energy density at the present epoch.

The analysis of the angular power spectrum of the cosmic microwave background implies a spatially flat Friedmann–Robertson–Walker universe [102]. For such a scenario, the scale factor a and the Hubble rate H are connected via the usual definition $H(t) := \dot{a}/a = \sqrt{8\pi G\rho(T)/3} =: H(T)$, where $\rho(T)$ is the energy density of all the matter at a

[33] See, e.g., section 9.2 of [99] for the physical meaning of each of the terms in equation (10.1). More detailed textbook derivations with a view towards various involved assumptions can be found, for instance, in [100, 101].

[34] We qualify the statement regarding CPT with the following observations: for the Standard Model fields $(CPT)^2 = +\mathbb{I}$, while for the unusual Wigner class under consideration, i.e., *Elko*, we have $(CPT)^2 = -\mathbb{I}$, as proven in section 4. This makes any statement on CPT a bit subtle. If one is considering purely the Standard Model physics CPT symmetry is fine. The same remains true if one is considering purely unusual Wigner classes. But as soon as one has both the standard and unusual Wigner fields present, the differing actions of CPT can induce a relative sign between the amplitude terms which involve only the Standard Model fields, and those involving an unusual Wigner class. However, as long as an amplitude does not contain the contraction of *Elko* with a Dirac field, no such relative sign comes into existence (for the quadratic appearances of Dirac fields, as well as *Elko* fields, the effect of $(CPT)^2$ is identical). So, at least for the process under consideration , i.e., $\varsigma + \stackrel{\neg}{\varsigma} \rightleftharpoons \phi_H + \phi_H$, CPT is respected. The general situation is likely to be more subtle and is expected to violate the CPT symmetry.

given temperature, and the overdot indicates a derivative with respect to time. Since non-relativistic matter components contribute negligibly, it is given by

$$\rho(T) = \frac{\pi^2}{30} T^4 \left[\sum_{i=bosons}{}' g_i + \frac{7}{8} \sum_{i=fermions}{}' g_i \right] := g_*(T)\, \frac{\pi^2 T^4}{30}\,, \tag{10.2}$$

where g_i represents helicity degrees of freedom associated with a specific component in the cosmic plasma, and the primed summation sign means that the summation is confined to those particles which may be considered relativistic at the relevant temperature, T. At temperatures of the order of $1 - 100$ MeV, the contributions arise from photons ($g_\gamma = 2$), three flavors of neutrinos and associated antineutrinos ($g_\nu = 6$), and electrons and positrons ($g_{e^\pm} = 4$). This yields $g_*(1 - 100 \text{ MeV}) = 10.75$. If there is a relativistic *Elko* component in the indicated point in the temperature range then $g_*(1 - 100 \text{ MeV}) = 14.25$.

Now, multiplying and dividing the factor of $\left(n_\varsigma a^3\right)$ on the left-hand side of equation (10.1) by T^3, and taking note of the fact that T roughly scales as a^{-1}, allows us to move $(aT)^3$ outside the derivative. This yields

$$T^3 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{n_\varsigma}{T^3}\right) = \left\langle \sigma_{\varsigma\,\bar{\varsigma} \to \phi_H \phi_H} |\mathbf{v}_{M\o l}| \right\rangle \left\{ \left(n_\varsigma^{\mathrm{EQ}}\right)^2 - n_\varsigma^2 \right\}. \tag{10.3}$$

Now define two new dimensionless variables, $Y := n_\varsigma / T^3$, and $x := m_\varsigma / T$, as well as

$$H(m_\varsigma) := \sqrt{\frac{4\pi^3 G\, g_*(m_\varsigma)\, m_\varsigma^4}{45}}\,, \quad \lambda_\varsigma := \frac{m_\varsigma^3}{H(m_\varsigma)} \left\langle \sigma_{\varsigma\,\bar{\varsigma} \to \phi_H \phi_H} |\mathbf{v}_{M\o l}| \right\rangle .$$

That done, the evolution equation becomes

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = \frac{\lambda_\varsigma}{x^2} \left\{ Y_{\mathrm{EQ}}^2 - Y^2 \right\} . \tag{10.4}$$

For a representative $m_\varsigma = 20$ MeV *Elko* particles, and taking $\langle \sigma_{\varsigma\,\bar{\varsigma} \to \phi_H \phi_H} |\mathbf{v}_{M\o l}| \rangle = 1$ pb we find $H(m_\varsigma) = 1.8 \times 10^{-22}$ GeV. Thus[35]

$$\lambda_\varsigma = 1.2 \times 10^8 \quad \left[\text{for } \left\langle \sigma_{\varsigma\,\bar{\varsigma} \to \phi_H \phi_H} |\mathbf{v}_{M\o l}| \right\rangle = 1 \text{ pb},\ m_\varsigma = 20 \text{ MeV} \right] . \tag{10.5}$$

Given $\lambda_\varsigma \gg 1$, the $Y(T)$ tracks Y_{EQ} at early times, i.e., for $1 < x \ll x_{\mathrm{f}}$ (where superscript/subscript f generically represent *freeze out*). For late times, i.e., $x \gg x_{\mathrm{f}}$, $Y(T)$ tracks $Y_{\mathrm{EQ}}(T)$ very poorly, and $Y(T) \gg Y_{\mathrm{EQ}}(T)$. Under this circumstance, the evolution of $Y(T)$ is determined by

$$\frac{\mathrm{d}Y}{\mathrm{d}x} \approx -\frac{\lambda_\varsigma}{x^2} Y^2, \quad (x \gg x_{\mathrm{f}}) . \tag{10.6}$$

Integrating this equation from the freeze-out epoch $x = x_{\mathrm{f}}$ to very late times $x = \infty$ (i.e., when all annihilations into Higgs have become extremely rare), and using the fact that on

[35] 1 pb $= 2.5681 \times 10^{-9}$ GeV^{-2}.

physical grounds $Y_\mathrm{f} \gg Y_\infty$ (i.e., *Elko* annihilations in evolution from x_f to x_∞ deplete ς and $\stackrel{\neg}{\varsigma}$), we get

$$Y_\infty \approx \frac{x_\mathrm{f}}{\lambda_\varsigma}\,. \tag{10.7}$$

A good analytical estimate for x_f can be obtained as follows. Define the departure from equilibrium by $\Delta := Y - Y_\mathrm{EQ}$, and use equation (10.4) with the substitution $Y^2 - Y_\mathrm{EQ}^2 = \Delta\,(2Y_\mathrm{EQ} + \Delta)$. This results in

$$\frac{\mathrm{d}\Delta}{\mathrm{d}x} = -\frac{\lambda_\varsigma}{x^2}\,\Delta\,(2Y_\mathrm{EQ} + \Delta) - \frac{dY_\mathrm{EQ}}{dx}\,. \tag{10.8}$$

Now at freeze out $x = x_\mathrm{f}$ corresponds to Y ceasing to track Y_EQ; that is, at freeze out $\Delta(x_\mathrm{f}) \simeq Y_\mathrm{EQ}(x_\mathrm{f})$. Substituting this into equation (10.8), and re-arranging yields

$$\frac{2}{Y_\mathrm{EQ}(x)}\frac{\mathrm{d}Y_\mathrm{EQ}(x)}{\mathrm{d}x}\bigg|_{x=x_\mathrm{f}} = -\frac{\lambda_\varsigma}{x_\mathrm{f}^2}\,3Y_\mathrm{EQ}(x_\mathrm{f})\,. \tag{10.9}$$

Using $Y_\mathrm{EQ}(x) = A\,x^{3/2}\,e^{-x}$ (for $x \geq 3$), with[36] $A := (45/2\pi^4)\sqrt{\pi/8}\,g/g_*(m_\varsigma)$, the left-hand side of equation (10.9) reduces to $-2 + (3/x_\mathrm{f})$. For $x_\mathrm{f} \gg 1$ (to be verified below), this factor takes the value of ≈ -2, resulting in

$$Y_\mathrm{EQ}(x_\mathrm{f}) = 2\left(\frac{x_\mathrm{f}^2}{3\,\lambda_\varsigma}\right)\,. \tag{10.10}$$

This gives (cf equation (5.43) of [101])

$$x_\mathrm{f} \approx \frac{1}{\sqrt{2}}\Big[\ln\left[3\,\lambda_\varsigma A\right] - 0.5\ln\left\{\ln\left[3\,\lambda_\varsigma\,A\right]\right\}\Big]\,. \tag{10.11}$$

Using the representative value of λ_ς given in equation (10.5), equation (10.11) yields $x_\mathrm{f} \approx 9.7$. In conjunction with equation (10.7) this implies that a rough estimate of relic abundance of *Elko* is $Y_\infty \approx 8.5 \times 10^{-8}$. For a 20 MeV *Elko* particle, at freeze out the temperature is $T_\mathrm{f} \approx 2.1$ MeV. Given this result we envisage that the asymptotic value Y_∞ is reached in an epoch when *no* significant reheating of the cosmic plasma occurs. That is, by $T_\mathrm{f} \approx 2.1$ MeV all the quarks and leptons, except $e^\pm$, have already annihilated and heated the photons in the cosmic plasma. Since $T_\mathrm{f} \approx 2.1$ MeV is very close to the boundary of where a slight reheating by $e^\pm$ may occur, and since $T_\infty \ll T_\mathrm{f}$, this justifies the working assumption of no significant reheating of the cosmic plasma after T_∞ is reached. For this reason

$$\left(\frac{a_\infty T_\infty}{a_0 T_0}\right)^3_\varsigma \approx 1\,, \tag{10.12}$$

where the subscript ∞ indicates the respective values in the epoch when Y reaches its asymptotic value, and the subscript 0 represents the present epoch. This contrasts dramatically with a heavy WIMP scenario, where significant reheating takes place, and

[36] Here g represents the number of helicity degrees of freedom of *Elko*, while $g_*(m_\varsigma)$ is roughly 10.75.

$(a_\infty T_\infty / a_0 T_0)^3_{\mathrm{WIMP}} \approx \frac{1}{30}$. Therefore, the *Elko* energy density *today* is

$$\rho_\varsigma = \left(\frac{a_\infty T_\infty}{a_0 T_0}\right)^3 m_\varsigma Y_\infty T_0^3 \approx m_\varsigma Y_\infty T_0^3 \tag{10.13}$$

For the representative case considered above, for the present epoch characterized by $T_0 = 2.725$ K (1K $= 8.617 \times 10^{-14}$ GeV) we find

$$\rho_\varsigma = 2.2 \times 10^{-47}\ \mathrm{GeV}^4 \quad \left[\text{for } \left\langle \sigma_{\varsigma\,\neg\varsigma \to \phi_H \phi_H} |\mathbf{v}_{M\o l}| \right\rangle = 1\ \mathrm{pb},\ m_\varsigma = 20\ \mathrm{MeV}\right]. \tag{10.14}$$

Since the critical density today, assuming $h = 0.72$, is $\rho_{cr} = 4.20 \times 10^{-47}$ GeV^4, the obtained result for *Elko* contribution to dark matter energy density today is very encouraging. This means that by choosing m_ς around 20 MeV and $\left\langle \sigma_{\varsigma\,\neg\varsigma \to \phi_H \phi_H} |\mathbf{v}_{M\o l}| \right\rangle$ around 1 pb one can readily obtain $\Omega_\varsigma \stackrel{\text{def}}{=} \rho_\varsigma / \rho_{cr} \approx 0.3$. To see this explicitly, the above formalism immediately yields

$$\Omega_\varsigma = \frac{1.89 \times 10^{-2}}{\langle \sigma v \rangle_{\mathrm{pb}}} \Big[2 \ln \left(9.31 \times 10^5 \langle \sigma v \rangle_{\mathrm{pb}}\, m_\varsigma^{\mathrm{MeV}} \right) - \ln \left[\ln \left(9.31 \times 10^5 \langle \sigma v \rangle_{\mathrm{pb}}\, m_\varsigma^{\mathrm{MeV}} \right) \right] \Big], \tag{10.15}$$

where $m_\varsigma^{\mathrm{MeV}}$ is the *Elko* mass in *MeV*, while $\langle \sigma v \rangle_{\mathrm{pb}}$ is the $\left\langle \sigma_{\varsigma\,\neg\varsigma \to \phi_H \phi_H} |\mathbf{v}_{M\o l}| \right\rangle$ in *picobarns.* Ω_ς carries a strong dependence on $\langle \sigma v \rangle_{\mathrm{pb}}$, and only a much weaker dependence on $m_\varsigma^{\mathrm{MeV}}$. For $\langle \sigma v \rangle_{\mathrm{pb}} = 2$, equation (10.15) gives $0.25 \le \Omega_\varsigma \le 0.33$ as the *Elko* mass is varied in the range $1 \le m_\varsigma^{\mathrm{MeV}} \le 100$. The value $\Omega_\varsigma = 0.30$ is obtained for $m_\varsigma^{\mathrm{MeV}} = 20$. The latter value for m_ς not only agrees with the considerations presented in section 10.2, but it is also supported by observations on the 0.511 MeV gamma-ray line made by the European Space Agency's INTEGRAL gamma-ray satellite [94–98]. The emergent relevant cross-section is also similar to the one suggested by Fayet's analysis for light dark matter [98]. As shown in section 10.2, for the discussed mass range, the collapse of a primordial *Elko* clouds also leaves a significant dark matter core at the center of galaxies.

The sensitivity of Ω_ς on $\langle \sigma v \rangle_{\mathrm{pb}}$ can be further gauged from the observation that for a $\langle \sigma v \rangle_{\mathrm{pb}} = 1$, equation (10.15) gives $0.47 \le \Omega_\varsigma \le 0.64$ for the same range of *Elko* mass as above, i.e., $1 \le m_\varsigma^{\mathrm{MeV}} \le 100$.

The $\sigma_{\varsigma\,\neg\varsigma \to \phi_H \phi_H}$ that appears in the above estimate of ρ_ς should be evaluated using the perturbative procedure outlined in section 8.3. Such an evaluation may provide additional insights and novel features which are not apparent from the presented calculations of this section. That $\varsigma\ \neg\varsigma \to \phi_H \phi_H$ cross-section will depend quadratically on λ_E^2 (see equation (9.3)) is evident. What is not clear is what precise contribution the *Elko* non-locality will make. This is perhaps the most important distinction from the WIMP scenario, and it remains to be studied in detail.

In order that Ω_{WIMP} is roughly 0.3, in a generic WIMP scenario the characteristic temperature for production and annihilations of WIMPs is put in by hand to be around a few hundred GeV. The thermal contact of *Elko* with the Standard Model cosmic plasma is severely restricted by the mass dimension one aspect. This introduces a characteristic

temperature below which the production and annihilations of *Elkos* are severely suppressed. This temperature is determined by the Higgs mass. Therefore, *Elko* particles constitute not only a first-principle candidate for dark matter, but also carry with them the property which dictates the above-mentioned characteristic temperature[37].

10.2 Collapse of a primordial Elko cloud: independent constraint on the Elko mass, $10^6\, M_\odot$ dark matter central cores for galaxies, and cosmic gamma-ray bursts

Here we consider a possible scenario which gives rise to the virialized dark matter clouds which overlap with luminous, standard-model, galaxies. The overlap is assumed purely on observational grounds, and we provide no *a priori* justification for this circumstance. We first give a brief run-through, and then proceed with the details.

10.2.1 A brief run-through

Schematically, we study a galactic-mass primordial ς and $\stackrel{\neg}{\varsigma}$ cloud which undergoes the following set of qualitative transformations:

$$P_{\varsigma\stackrel{\neg}{\varsigma}} \rightarrow R_{\varsigma\stackrel{\neg}{\varsigma}} \rightarrow V_{\varsigma\stackrel{\neg}{\varsigma}}\,. \tag{10.16}$$

Here,

- $P_{\varsigma\stackrel{\neg}{\varsigma}}$ represents a primordial ς- $\stackrel{\neg}{\varsigma}$ cloud. Its spatial extent is assumed to be a few times that of a typical galaxy.

- $R_{\varsigma\stackrel{\neg}{\varsigma}}$ is a rebound caused (a) either by *Elko's* quartic self-interaction (requiring cloud temperature of about $T_\varsigma \sim mc^2/k_{\mathrm{B}}$), or (b) by the *Elko-Elko* interaction producing pair of Higgs (requiring cloud temperature of about $T_{\mathrm{H}} \sim m_{\mathrm{H}}c^2/k_{\mathrm{B}}$, with m_{H} as the Higgs mass). The possibility that rebounds occurs due to some quantum gravity effect is also considered.

- $V_{\varsigma\stackrel{\neg}{\varsigma}}$ is the virialized *Elko* cloud which emerges after the above-indicated rebound and thermalization-inducing process.

The scenario in which the rebound of a primordial *Elko* cloud is induced by the process $\varsigma+ \stackrel{\neg}{\varsigma} \rightleftharpoons \phi_H + \phi_H$, carries three basic results: (a) it sets a lower bound of about 1 MeV for m_ς (called simply m in section 10.1), (b) it suggests $10^6\, M_\odot$ dark matter central cores for typical galaxies, and (c) it predicts an explosive event in the early life history of galaxy formation.

The physical criterion that provides the above-enumerated results is the requirement that T_ς, or T_{H} (or T_{QG}, associated with the Planck scale) is reached before the cloud radius crosses the Chandrasekhar limit. If this requirement is not met, then one ends up with a degenerate *Elko* core, or a black hole. These may, or may not, have association with luminous galaxies.

[37]This section was added on suggestion from a *JCAP* referee.

Some elements of our exercise are textbook like. Yet, the requirement just enunciated yields a rich set of results. As a parenthetical remark, it is emphasized that we have not encountered these considerations in literature on dark matter[38]. It is probably due to the fact that the lower mass bound derived below is often superseded by stricter bounds; for example, in the context of SUSY dark matter the lower mass bounds on the lightest SUSY particle are well within the 10–100 GeV range. These, however, do not apply to MeV-range Elko mass.

10.2.2 Details

For simplicity we assume a spherical distribution, characterized by mass M (of the order of a typical galactic mass), and initial radius R, of ς particles undergoing a gravitationally induced collapse. The cloud continues to collapse, and its temperature soars until it reaches a temperature $T_* \sim m_* \, c^2/k_{\rm B}$, where '$*$' characterizes either of the two mass scales which appear in the additional *Elko*–induced structure of the Standard Model Lagrangian density (see equations (9.1)–(9.3))[39]. At that stage the *Elko* cloud has the possibility to radiate electromagnetically and/or to emit neutrinos, and to cool down *provided T_* is reached when the spatial extent of the collapsing ς cloud is greater than the Chandrasekhar limit, R_{Ch}* —a condition which will be examined below (and roughly signals black hole formation). Under these circumstances, the newly available radiation pressure may cause part of the cloud to explode, while leaving an imploding remanent fated to either become a black hole or a degenerate *Elko* core, the *Elko* analogue of a neutron star. If such a scenario is to explain the dark matter problem the exploding *Elko* envelope must, at present, carry dimensions of the order the galactic size. This is not a prediction, but what detailed calculations should yield if the presented scenario is to be viable as a dark matter candidate. In essence for this to happen a substantial fraction of the released explosive energy must be re-deposited to the expanding ς envelope.

To determine the mass and size of such a remanent dark matter configuration we make the working assumption that non-locality plays insignificant role. Then, the standard arguments of balancing the Fermi pressure against the gravitational potential energy yield the following critical Chandrasekhar values:

$$M_{\rm Ch} \approx \left(\frac{m_{\rm P}}{m}\right)^3 m\,, \quad R_{\rm Ch} \approx \left(\frac{m_{\rm P}}{m}\right) \lambda_{\rm C}\,, \tag{10.17}$$

where m is ς mass, $\lambda_{\rm C}$ is the associated Compton length, $\hbar/mc$, and $m_{\rm P} := \sqrt{\hbar c/G}$ is the Planck mass. The set $\{M_{\rm Ch},\ R_{\rm Ch}\}$ sets the boundary between stability and instability and also marks the complete onset of general relativistic effects. To proceed further it is essential to gain some knowledge of plausible values of $M_{\rm Ch}$ and $R_{\rm Ch}$. For this we recast $M_{\rm Ch}$ and $R_{\rm Ch}$ in the following form:

$$M_{\rm Ch} \approx \frac{1}{x_\varsigma^2}\, 1.6\times 10^{12} M_\odot\,, \quad R_{\rm Ch} \approx \frac{1}{x_\varsigma^2}\, 6.3\times 10^{-2}\ {\rm pc}\,, \tag{10.18}$$

[38] After this work was submitted to *arXiv* — see *v1* of the present work [103] — the approach taken here has been independently adopted by Vanderveld and Wasserman [104].

[39] In this section we exhibit $\hbar$, c, $k_{\rm B}$, G explicitly.

where $M_\odot$ is the solar mass, and x_ς represents the *Elko* mass m expressed in keV.

With these observations in mind, we conjecture that the collapse physics of such a cloud is similar to that of a supernova explosion. It leaves behind a degenerate structure of ς particles, or a black hole with mass $M_{\rm Ch}$, while releasing $(M - M_{\rm Ch})$ as a sum total for the mass of rebounding ς envelope, M_ς, and an energy burst — predominantly made of gamma rays, and neutrinos — carrying $M_{\gamma,\nu}\, c^2$. We shall assume that the coupling constants α_E, λ_E, and their interplay with gravity, are such that the *Elko* cloud does not develop significant density fluctuations to seed — over the timescale of its collapse — the formation of smaller structures.

We now examine the condition for which T_* is reached when the spatial extent of the collapsing ς cloud is greater than the Chandrasekhar limit, $R_{\rm Ch}$. Let R_* represent the radius which characterizes the spatial extent of the collapsing ς cloud when it reaches the temperature T_*. Under the assumption that the initial $R \gg R_*$, R_* is characterized by the *Elko* configuration when the average kinetic energy gained per ς equals the energy associated with a Higgs or ς (the only two mass scales which *Elko*-induced new Lagrangian density carries)[40]:

$$\frac{GM^2}{NR_*} = m_* c^2 \,, \tag{10.19}$$

where $N = M/m$ is the number of ς particles in the cloud, while m_* is either the ς mass m, or it represents the Higgs mass m_H. Taking note of the fact that $m_{\rm P}^3\, m_{\rm p}^{-2}$ is a typical stellar mass ($= 3.77 \times 10^{33}$ g $\approx 1.9 M_\odot$), we write $M \approx M_{\rm G} \approx \alpha_{\rm star}\, m_{\rm P}^3\, m_{\rm p}^{-2}$, where $M_{\rm G}$ represents the luminous mass of a typical galaxy, α_{star} is approximately the number of stars in the same, and m_p is the proton mass, and inserting for R_* the absolute lower bound of $R_{\rm Ch}$, equation (10.19) yields

$$\alpha_{\rm star} = \frac{m_{\rm p}^2\, m_*}{m^3} \,. \tag{10.20}$$

This remarkable equation can be read in two ways: with the m_*/m^3 as input a rough estimate for the number of stars in a typical galaxy may be derived; on the other hand, with $\alpha_{star} \approx 10^{11}$ as input, one may obtain the ratio m_*/m^3. Since observationally $\alpha_{star} \approx 10^{11}$ is known, this immediately implies the following results for the mass of the ς particles:

$$m = \begin{cases} \frac{m_{\rm p}}{\sqrt{\alpha_{\rm star}}} & \text{for } m_* = m \,, \\ \left(\frac{m_{\rm p}^2\, m_{\rm H}}{\alpha_{\rm star}}\right)^{1/3} & \text{for } m_* = m_{\rm H} \,. \end{cases} \tag{10.21}$$

Taking a representative value of $\alpha_{\rm star} = 10^{11}$, and writing the Higgs mass $m_{\rm H} = x_{\rm H}\, 100$ GeV$/c^2$, with $1 \leq x_{\rm H} \leq 2$, the above equation yields

$$m = \begin{cases} 3 \text{ keV}/c^2 & \text{for } m_* = m \,, \\ 1.0 - 1.2 \text{ MeV}/c^2 & \text{for } m_* = m_{\rm H} \,. \end{cases} \tag{10.22}$$

[40]We do not include a factor of 2 on the rhs of equation (10.19) as all our calculations in this section carry order-of-magnitude estimates.

These values of m are to be considered as lower bounds because the requirement one has to impose is $R_* \geq R_{\rm Ch}$, while (10.20) was obtained by saturation of this inequality.

To decide between these two values of the *Elko* mass, we now make the observation that

$$M_\varsigma + M_{\gamma,\nu} := M - M_{\rm Ch} = \left(1 - \frac{m_{\rm p}^2}{\alpha_{\rm star}\, m^2}\right) \frac{\alpha_{\rm star}\, m_{\rm P}^3}{m_{\rm p}^2}\,. \tag{10.23}$$

On the extreme rhs of the above equation, identifying the factor outside the bracket as M and using equation (10.20) inside the bracket, we get

$$M_\varsigma + M_{\gamma,\nu} = \left(1 - \frac{m}{m_*}\right) M\,. \tag{10.24}$$

Since without a rebound of the *Elko* cloud the viability criterion cannot be met[41], the sum $M_\varsigma + M_{\gamma,\nu}$ must be a *good* fraction of M. For this to occur, the round bracket in (10.24) must not become too small. For the solution with $m_* \approx m_{\rm H}$, the central degenerate core (or, a black hole) carries a mass of about $10^6\, M_\odot$ while rebounding *Elko* cloud, and the associated burst of energy, carries almost the entire mass. For $m \approx m_*$, the mass of the remanent core becomes of the order of M itself. That one cannot make a more precise statement for the mass of the rebounding cloud near $m_* = m$ is due to the order-of-magnitude nature of the calculation and may be considered as a drawback (there is no prediction from order-of-magnitude estimates) or as a virtue (the rebounding cloud is very sensitive to Elko details and thus may probe Elko physics). Unless m is very finely tuned the round bracket in (10.24) is still of the order of unity and thus again the rebounding *Elko* cloud carries a mass of the order of M[42]. For $m_* \gg m$ the order-of-magnitude estimate is more robust.

In *favour* of the solution $m \approx m_*$ one is tempted to note that, apart from the dark matter problem, there are two outstanding cases where the fermions of the Standard Model of particle physics fail to provide the astrophysical consequences expected of them. These are pulsar kicks [105, 106] and supernova explosions [107]. Kusenko [105, 106], and Dolgov and Hansen [108] have argued that a sterile component with a mass of about 2–20 keV/c^2 provides a good candidate to explain pulsar kicks and that the same particle may also be cosmological dark matter. However, for the argument to work for pulsar kicks it is essential that the sterile component must carry an intrinsic parity asymmetry. Such an asymmetry is naturally built in the *Elko* particles, ς and $\stackrel{\neg}{\varsigma}$. But, as long as one confines oneself to the set of assumptions we have used, the 4.2 keV/c^2 identification noted here gives rise to too high a value for the galactic core. It thus violates the minimal viability criteria. In addition, there seems to be a more fundamental problem with such an identification: how is one to add a mass dimension one component (Elko) to a mass dimension three-halves field

[41] That is, observationally, the mass of the dark matter cloud should be of the order of a typical galactic mass and its spatial extent must extend beyond luminous extent of galaxies. This we call the minimal viability criteria for *Elko* to be a dark matter candidate.

[42] In other words, for $m_* \approx m$ the $M_{\rm Ch}$ becomes of the order of the initial mass of the *Elko* cloud M itself, and hence its difference from the latter ceases to have a reliable meaning.

(neutrinos)? The problem appears to be non-trivial and has no clear answer due to subtle questions which mixing of local and non-local field on the one hand, and mass dimension three-halves and one on the other, raises.

In *favour* of the solution $m \approx m_{\mathrm{H}}$ we have the additional support from two recent works. From observations on 0.511 MeV/c^2 gamma-ray line seen by the European Space Agency's INTEGRAL gamma-ray satellite [109], Boehm *et al* [94, 95], followed by additional observations of Beacom *et al* [96], purport to read a dark matter particle mass in the range of 1–20 MeV.

The $m_* = m_{\mathrm{H}}$ identification meets the minimal viability criteria well. Yet, the strength of this viability and identification must not be overestimated. There are several open points regarding these considerations.

- Quartic Elko self-interactions have been neglected. These may contribute in an essential way in the energy range considered. Moreover, for stability reasons the coupling constant in front of this term has to be such that a repulsive interaction emerges. As a result this may contribute to the rebound of the collapsing Elko cloud.
- Non-locality has been neglected, but because it scales essentially with m^{-1} contributions from it could be relevant here.
- Non-standard gravitational interactions could be of importance in this context – for instance, the 'square' of *Elko* field, which is a scalar of mass dimension two, may couple to the Ricci scalar with a dimensionless coupling constant, much like the Jordan–Brans–Dicke field [110, 111] does in scalar tensor theories or quintessence models [112–115].

Therefore, our simple order-of-magnitude considerations above cannot be used to determine uniquely the mass of Elko, but it appears to be likely that it lies in the range of 1 keV/c^2 to 20 MeV/c^2 if *Elko* is to explain dark matter. It is desirable to improve these limits further, but to this end one has to address the caveats mentioned above in a more detailed study, possibly combining the two scenarios with critical temperatures of T_H and T_ς, respectively.

In conclusion, the gravitationally induced collapse of an *Elko* cloud, if ς and $\stackrel{\neg}{\varsigma}$ are to serve as dark matter, must be qualitatively similar to type-II supernova explosions of stellar objects which leave a degenerate core of fermionic matter and are accompanied by (a) an expanding envelope of matter, and (b) an intense electromagnetic and neutrino radiation carrying several solar masses. For an *Elko* cloud, the neutronic core is replaced by a massive degenerate core of ς (or, a black hole) — indicated here to be of the order of 10^6 solar masses — while the gamma-ray and neutrino radiation may carry a mass of the order of galactic mass and the burst temperature T_* may be characterized by two characteristic masses: the Higgs mass and the mass of ς.

Given that we are confronted with a truly unknown cosmic phenomenon, it is not beyond reason that the following sequence of multiple rebounds is realized. In that event one can no longer reply on the one-mass scale dominance scenario outlined above. Therefore,

no quantitative analysis can be presented at this stage of our work. In this scenario, if one assumes that the Electroweak and GUT fields do not carry any significant coupling to *Elko* then only one additional mass scale comes into the picture, and it is given by $m_{\rm P}$. As the collapsing *Elko* cloud successively soars to this temperature one may expect that apart from an explosion induced by quartic self-interaction of *Elko* itself another explosive phase occurs at the Higgs temperature, and an unknown quantum-gravity induced effect occurs at the Planck temperature. Should the latter carry an explosive element, and should the *Elko* self-interaction and Higgs-mediated explosions not succeed in causing a significant rebound, then a one-mass scale scenario predicts *Elko* mass to be

$$m = \left(\frac{m_{\rm p}^2\, m_{\rm P}}{\alpha_{\rm star}}\right)^{1/3} . \tag{10.25}$$

With $m_{\rm P} = 1.2 \times 10^{19}$ GeV$/c^2$, this results in the *Elko* mass being 0.5 TeV$/c^2$. In this scenario almost all of the *Elko* clouds rebounds without leaving a remanent core.

10.3 Similarities and differences from the mirror matter proposal

We recall that mirror matter, which restores parity symmetry by introducing a parallel universe, is postulated to be endowed with the following properties.

- Particle masses in the mirror world are set degenerate to the masses of the Standard Model particles of the world in which we reside.
- The mirror world carries same gauge group as that of the Standard Model, with all left-handed fields replaced by right-handed fields and vice versa.
- The mirror world has the following dominant interactions with the SM fields:

  $$\mathcal{L}(x) = \lambda_{\rm M}\, \phi^\dagger(x)\phi(x)\, {\phi'}^\dagger(x)\phi'(x) + \epsilon_{\rm M}\, F^{\mu\nu}(x)F'_{\mu\nu}(x) \tag{10.26}$$

  where $\phi(x)$ is the Standard Model Higgs doublet, and $F^{\mu\nu}(x)$ is the electromagnetic field strength tensor — with primed fields representing their mirror counterparts, while $\lambda_{\rm M}$ and $\epsilon_{\rm M}$ are dimensionless coupling constants.

It is a viable and attractive dark matter candidate [116–118]. Over the years it had been suspected that mirror matter, apart from being a dark matter candidate, could also provide a resolution to the orthopositronium life time puzzle which had acquired a statistical significance well above the 5-σ level. Latest measurements by two independent groups, however, have discovered a systematic source of errors and now these latest experiments provide a decay rate which agrees well with predictions of quantum electrodynamics [119, 120].

The *Elko* proposal for dark matter has some noteworthy similarities and differences from the mirror matter framework. The similarity between equations (9.3) and (9.4), and the mirror counterpart just enumerated, is obvious, and carries its roots in mass dimension one of the *Elko* field. Yet, apart from the assumption that standard power-counting renormalizability arguments provide a good guide, in case of *Elko* no postulate

has to be made to restrict the interactions with the Standard Model matter to the specific form.

Mirror symmetry predicts mirror stars, and claims an indirect support for their existence [121,122]. Yet, we suspect these claims for the simple reason that the evidence must have been overwhelming already because mirror stars would have already been observed as numerous dark companions of luminous SM-model stellar objects *unless* one invokes some arguments as to why the dark matter and luminous matter of galaxies occupy overlapping spatial regions and yet the indicated circumstance does not seem to arise.

One of advantages of our proposal, which to some extent is a subjective perception, is that the *Elko* framework does not have to invoke a parallel universe. It thus avoids the inevitable formation of mirror stars, and mirror planets — which are contained in the framework of unbroken mirror symmetry. Broken mirror symmetry does not seem to be a way out as it destroys the very motivation upon which the mirror symmetry is based. Yet, an interesting possibility with parity symmetry being spontaneously broken has been suggested by Berezhiani and Mohapatra [123].

On the other hand, the mirror–matter framework is local at a quantum field theoretic level. The *Elko* framework carries a well defined non-locality, whose physics motivation have been discussed at some length in [21].

These remarks aside, both mirror matter and *Elko* matter provide most natural candidates for dark matter without invoking any untested spacetime symmetries. For this reason, and due to their similarities, it should be considered as an urgent task to distinguish their implications for astroparticle physics, cosmology, and experiments such as those envisaged for mirror matter [124]. While pursuing these studies, the following observations should be kept in mind.

- The fermions of the mirror matter carry mass dimension three-halves. In contrast, the fermion[s] of the *Elko* matter carry mass dimension one.

- The limited interactions which the *Elko* matter carries with itself, and with the Standard Model fields, are dictated by its mass dimension. At present, is not known what effects the intrinsic *Elko* non-locality carries. The mirror matter postulates the absence of a large class of possible interactions with the Standard Model fields.

- Given differing mass dimensionalities of the mirror and *Elko* fermions, it is more natural that the suspected sterile component of neutrinos *may* be a mirror neutrino. Such a suggestion has been made in [123,125]. The introduction of an *Elko* fermion as a sterile component has not yet been investigated, due to subtle questions which non-locality and its mass dimension raises.

- Mirror matter demands the existence of mirror stars, planets, and even meteorites [126]. The minimal *Elko* proposal set forth here in this paper does not suggest any such objects.

11. Elko particles in a Thirring–Lense gravitational field

In the above discussion the role of $\overline{\varsigma}$ was entirely overlooked. Here, in this section, we now find that it may carry significance for the matter–antimatter asymmetry. We have not yet developed a full theory based on *Elko* in the gravitational background of a rotating gravitational source. Yet, an important physical implication has already been studied [127] in which Elko single-particle states are considered in Thirring–Lense gravitational field[43]. This example is an important physical case where a preferred direction *does* exist and the more general structure of the spacetime evolution must be invoked.

The circumstance which allows for a 'back-of-the-envelope' calculation is the following: the $(1/2,0)$ and the $(0,1/2)$ Weyl components, for each of the *Elko*, carry opposite helicities. Based on the particle interpretation developed in this paper we expect single-particle states to carry the same property. Thus if we introduce a self-conjugate state $|+,-\rangle_{\mathrm{S}}$ in the gravitational environment of a rotating gravitational source, such as a neutron star, then each of the $(1/2,0)$ and $(0,1/2)$ transforming components of the state pick up equal and opposite phases (where we now display $\hbar$ explicitly):

$$(1/2,0): \quad \exp\left(-\,\frac{\mathrm{i}}{\hbar}\times\frac{\hbar}{2}\,|\widehat{\mathbf{p}}\cdot\mathbf{b}|\,t\right) \tag{11.1}$$

$$(0,1/2): \quad \exp\left(+\,\frac{\mathrm{i}}{\hbar}\times\frac{\hbar}{2}\,|\widehat{\mathbf{p}}\cdot\mathbf{b}|\,t\right)\,, \tag{11.2}$$

with $\mathbf{b}$ representing the Thirring–Lense field of the source star. In the weak field limit it is given by[44]

$$\mathbf{b} = \frac{2G}{c^2}\left(\frac{\boldsymbol{\mathcal{J}} - 3\left(\boldsymbol{\mathcal{J}}\cdot\widehat{\mathbf{r}}\right)}{r^3}\right)\,. \tag{11.3}$$

Here G is the Newtonian gravitational constant, c is the usual speed of light, $\boldsymbol{\mathcal{J}}$ is the angular momentum of the rotating gravitational source (denoted by $\mathcal{G}$), and r specifies the radial coordinate distance of the region where the particle is observed. The magnitude of $\boldsymbol{\mathcal{J}}$ is given by $\mathcal{J}\approx\frac{2}{5}MR^2\omega$, where M is the mass of $\mathcal{G}$, R is its radius, and ω represents the associated angular frequency. A couple of observations are immediately warranted.

- In phases (11.1) and (11.2), $\hbar$ cancels out. This happens because the $\hbar$ which appears in a generic quantum evolution and the $\hbar$ which appears in a particle's spin/helicity cancel out.

- In (11.1) and (11.2), t refers to the ensemble-averaged time as measured by a stationary clock situated in the gravitational environment at $\mathbf{r}$. The qualification, ensemble-averaged, is necessary for generalized flavour-oscillation clocks [128, 129].

[43] In the context of [127] the reader is alerted that the identification of the described particles with neutrinos and antineutrinos made there, in view of the work presented here, is no longer tenable. The correct identification is with ς and $\overline{\varsigma}$.

[44] This is a valid approximation all the way to a neutron star as the dimensionless gravitational potential $GM/c^2R\approx 0.2$ for a 1.4 solar mass neutron star.

The spinor associated with $|\pm,\mp\rangle_{\mathrm{S/A}}$ are $\lambda^{\mathrm{S/A}}_{\{\pm,\mp\}}(\mathbf{p})$. Using phases which the coupling of these states with the $\mathbf{b}$ field induces, we find the following oscillations[45]:

$$|\pm,\mp\rangle_{\mathrm{S}} \leftrightharpoons |\pm,\mp\rangle_{\mathrm{A}}\,, \tag{11.4}$$

with oscillation probability $\sin^2(\widetilde{\omega}_{\mathrm{osc.}}t)$, where

$$\widetilde{\omega}_{\mathrm{osc.}} = \frac{4}{5}\left(\frac{GM}{c^2R}\right)\omega\,. \tag{11.5}$$

Since in the presence of a preferred direction, the study of the wave equation for *Elko* associates particle nature to self-conjugate sector, and provides an antiparticle interpretation for the anti-self-conjugate sector of the *Elko*, we are led to conclude that rotations in gravitational environments induce $\varsigma \leftrightharpoons \stackrel{\neg}{\varsigma}$ oscillations, where ς and $\stackrel{\neg}{\varsigma}$ represent particles and antiparticles associated with the neutral $\eta(x)$ field. A sea composed entirely of ς will in time develop a $\stackrel{\neg}{\varsigma}$ component, thus inducing an *Elko* matter–antimatter asymmetry. Since we identify the ς and $\stackrel{\neg}{\varsigma}$ with the dark matter, these oscillations may have an important role to play in cosmology, although we hasten to add that the observed baryonic matter–antimatter asymmetry is unlikely to be a consequence of this because, after all, *Elko* interacts only very weakly with the matter content of the Standard Model.

12. A critique and concluding remarks

In this section we combine the customary concluding remarks with additional details of our work. This section, with exception of sections 12.5 and 12.6, is not a summary of our work but is intended as an integral part of our exposition. Its purpose is to bring to reader's attention additional structure which the *Elko*-based quantum field theory carries, and in addition to point out that locality is most likely not a stable feature of quantum field theories at the Planck scale. However, the level of rigour now changes and it mirrors the previous section, giving us the luxury of some speculative remarks.

12.1 Elko as a generalization of Wigner–Weinberg classes

That our effort had a chance of providing us with a theoretically viable and phenomenologically novel theory seemed assured by Wigner's work on the extended Poincaré group [6]. In that work a general result is reached that the Poincaré group extended with space, time, and space–time reflections allows every representation space to support four type of quantum field theories. Each of these theories differs in the underlying C, P, and T properties[46]. For the particles of the Standard Model, if $\mathcal{C}$ and $\mathcal{P}$ generically represent charge conjugation and parity operators, respectively, then

$$\text{STANDARD MODEL:}\qquad \begin{cases} \{\mathcal{P},\mathcal{C}\}=0, & \text{for fermions: leptons, quarks} \\ [\mathcal{P},\mathcal{C}]=0, & \text{for bosons: gauge/Higgs particles}\,. \end{cases} \tag{12.1}$$

[45] The calculation has been made under the assumption that we prepared the test particle in such a way that $\widehat{\mathbf{p}}=\widehat{\mathbf{r}}$, and that its evolution is studied in the polar region.

[46] The origin of these observations in fact lies in an unpublished work of Bargmann, Wightman, and Wigner. In [6] Wigner states 'Much of the material which follows was taken from a rather old but unpublished manuscript by V. Bargmann, A. S. Wightman, and myself'.

In [6], Wigner argues that the above class does not exhaust all possibilities allowed by Poincaré symmetries. In fact, he showed that the following additional structure is allowed:

$$\text{WIGNER CLASSES:} \qquad \begin{cases} [\mathcal{P}, \mathcal{C}] = 0, & \text{for fermions: the presented construct} \\ \{\mathcal{P}, \mathcal{C}\} = 0, & \text{for bosons: construct of [20]}\,. \end{cases} \tag{12.2}$$

Yet, for the sake of avoiding confusion we take note that the presented construct is a generalization of Wigner–Weinberg context [8]. Specifically, in the context of Weinberg's work, which in turn is an extension of Wigner's considerations, equation (2.C.10) of [8],

$$\mathcal{P}\,|\mathbf{p}, \sigma, n\rangle = \sum_m \wp_{nm}\,|-\mathbf{p}, \sigma, m\rangle \tag{12.3}$$

is no longer sufficiently general. In reference to the above equation, we give the following notational details: the $|\mathbf{p}, \sigma, n\rangle$ represent one-particle states, $\mathbf{p}$ is the momentum vector, $\mathcal{P}$ is the parity operator and carries properties defined on p 76 of [8], σ represents one of the $(2j+1)$ spin projections associated with J_3, and n, m define a degeneracy, and the $\wp_{nm}$ are superposition coefficients. In relevance to the theory which we present here, while the degeneracy index may be interpreted as associated with self-conjugate and anti-self-conjugate states, the counterpart of the σ, i.e., the dual-helicity index α, introduces additional structure and allows for the result obtained in section 4.2. This observation casts the *Elko* formalism into a further generalization of the Wigner–Weinberg framework, and while remaining in agreement with Wigner we provide a counter-example to Weinberg's result contained in equation (2.C.12) of [8]. The counter-example contained in the result (4.21) is not to be interpreted as a contradiction, but as a generalization.

12.2 On Lee–Wick non-locality, and Snyder–Yang–Mendes algebra

Lee and Wick have argued that the non-standard Wigner classes must carry an element of non-locality [7]. The *Elko*-based quantum field $\eta(x)$ exhibits a non-locality (8.20) which is consistent with this expectation. Yet, we cannot fully identify the non-local element which we discovered with that of Lee and Wick as our formalism is a generalization of the Wigner framework for which Lee–Wick considerations apply.

What, however, can be claimed is the following: locality cannot be considered a robust feature of Poincaré covariant quantum field theories. This is apparent from Lee–Wick paper as well as from the explicit construct which we present. The question then arises: why does this circumstance exist? The answer, we think, lies in the important reminder and observations of Mendes that [56, 130]

(a) The Galilean relativity and classical mechanics are unstable algebraic structures, i.e., in the mathematical sense they are not rigid.

(b) Their deformation towards stability results in Poincaré and Heisenberg algebras of special relativity and quantum mechanics. Each of the indicated algebras is stable by itself.

(c) The algebra for relativistic quantum field theory is the combined Poincaré–Heisenberg algebra, which is unstable.

The stabilized Poincaré–Heisenberg algebra as presented by Mendes still respects Lorentz algebra[47], but calls for non-commutative spacetime and non-commutative energy-momentum-space[48]. This introduces two stabilizing dimensionful constants which may be identified with Planck length and cosmological constant. It is worth taking note that in 1947 Yang had already argued that lack of translational invariance of Snyder's algebra, also suggested earlier in the same year, is remedied if spacetime is allowed to carry curvature [134, 135], and that the algebra found on algebra-stability grounds by Mendes is precisely the one that Yang had arrived at more than five decades ago.

A preliminary examination of the Snyder–Yang–Mendes algebra suggests that the usual expectation on locality commutators/anticommutators cannot hold. We phrase it as 'locality is not a stable feature of quantum field theories', and suspect that it will completely lose its conventional meaning in quantum field theories based on stabilized Poincaré–Heisenberg algebra. Following Mendes, that stability of algebra associated with a physical theory should be considered as an important physical criterion for the viability of a theory has also been emphasized by Chryssomalakos [131, 136, 137].

Yet, just as Dirac theory will suffer modifications in any extension which respects the stabilized Poincaré–Heisenberg algebra, i.e., the Snyder–Yang–Mendes algebra, but still remains a useful low-energy model, the same may be expected for the *Elko* theory presented here. It may serve as a low-energy model to explore consequences of non–locality.

12.3 A hint for non-commutative momentum space

Evaluating the determinant of the operator $\mathcal{O}$ in (5.33) with (2.2), (2.3) and (3.1) establishes

$$Det[\mathcal{O}] = \frac{\left(m^2+p^2-(2m+E)^2\right)^2 \;\left(m^2+p^2-E^2\right)^2}{\left(2m(E+m)\right)^4}\,. \tag{12.4}$$

The momentum–space wave operator, $\mathcal{O}$, *appears* to support two type of spinors: those associated with the usual dispersion relation,

$$E^2 = m^2+p^2\,, \qquad \text{multiplicity} = 2 \tag{12.5}$$

and those associated with

$$E = \begin{cases} -\,2m - \sqrt{m^2+p^2}\,, & \text{multiplicity} = 1 \\ -\,2m + \sqrt{m^2+p^2}\,, & \text{multiplicity} = 1\,. \end{cases} \tag{12.6}$$

[47] For a detailed mathematical and interpretational examination of the stability-based framework of Mendes, we bring to our reader's attention a recent preprint by Chryssomalakos and Okon [131].

[48] It should be pointed out that the idea that quantization of gravity leads to non-locality and/or non-commutative spacetime has various roots; see, for example, [22, 23] and references therein. In some simple models — for instance, dilaton gravity in two dimensions — these ideas turn out to be realized straightforwardly, i.e., quantization of gravity yields a non-local theory for the remaining matter degrees of freedom (for reviews [132, 133] may be consulted). Arguments based on incorporating gravitational effects in a quantum measurement process also suggest an element of spacetime non-commutativity and non-locality [55, 57].

However, it is now to be noted that the Det $[\mathcal{O}]$ is independent of $\mathcal{A}$, and that dispersion relations (12.6) yield

$$\kappa^{(1/2,0)}\kappa^{(0,1/2)}\Big|_{E=-\,2m\pm\sqrt{m^2+p^2}} = \kappa^{(0,1/2)}\kappa^{(1/2,0)}\Big|_{E=-\,2m\pm\sqrt{m^2+p^2}} = -\mathbb{I}\,. \tag{12.7}$$

Obviously, (12.7) is inconsistent with the starting point as (2.2) is the inverse of (2.3). Thus, there are only two options: either the solutions (12.6) are discarded or the relation between $\kappa^{(1/2,0)}$ and $\kappa^{(0,1/2)}$ is modified. *It is conceivable that for example in the context of non-commutative momentum space the latter scenario might be realized.* This possibility is again taken up in appendix B.6.

An explicit calculation shows that[49]

$$\kappa^{(j,0)}\kappa^{(0,j)}\Big|_{E=-\,2m\pm\sqrt{m^2+p^2}} = \kappa^{(j,0)}\kappa^{(0,j)}\Big|_{E=-\,2m\pm\sqrt{m^2+p^2}} \neq -\mathbb{I}\,, \tag{12.8}$$

unless $j=\frac{1}{2}$. It is indicative of the fact that the above-mentioned non-commutativity is probably representation-space dependent .

12.4 Generalization to higher spins

It may be worth noting that a generalization of the $(1/2,0)\oplus(0,1/2)$ *Elko* field to higher $(j,0)\oplus(0,j)$ spins is perhaps not too difficult a conceptual exercise. If such an exercise is undertaken it may reveal that in the absence of a preferred direction *all* such higher spin fields will carry mass dimension one. This may have important consequence for the physical understanding of representation spaces which carry a single-spin j content.

For other representation spaces, such as spinor-vector Rarita–Schwinger field, an *ab initio* analysis along the lines of [67,68] may also be worth considering. In such an analysis, the spinor sector would have to be given an *Elko* structure, and the resulting propagator will almost certainly carry new physical content.

The purpose of these remarks is simply to emphasize that the physical content of various physically relevant representation spaces is far from complete, and a serious *ab initio* study appears to carry significant physical promise as to give such a task an element of urgency. This becomes even more justified when one realizes that various *exotica*, such as non-locality, non-commutative spacetime, modified dispersion relations, all arise in constructs which go far beyond the point-particle context. As such, the suggested programme may be considered as a minimal departure from the standard high-energy physics framework with a potential to allow a systematic study of the indicated *exotica*.

12.5 A reference guide to some of the key equations

For a reader interested in the main theoretical expressions we now provide a brief reference list. Equations (3.3)–(3.5) provide the formal structure of the *Elko*. Their dual-helicity nature follows from the result (3.8). Like the Dirac dual, there exists a new dual for *Elko*. It is defined in equation (3.19). With respect to this dual, the orthonormality and

[49]To be precise, we have verified the result (12.8) only for $j=1$ and $j=\frac{3}{2}$. As such for $j>\frac{3}{2}$, equation (12.8) should be considered a conjecture.

completeness relations are enumerated in equations (3.23)–(3.25). Wigner's expectation for commutativity, as opposed to *Diracian* anticommutativity, of the C and P operators while acting on *Elko* is contained in equation (4.16), while the square of the CPT operator as contained in equation (4.21) shows another difference from the Dirac construct. The *Elko* quantum field and its dual are given in equations (6.1) and (6.2). The action and the Lagrangian density for *Elko* are contained in equations (7.3) and (7.4). Equation (7.5) gives the canonically conjugate momentum. The covariant amplitude which determines the *Elko* propagator in terms of the *Elko* spin sums is found in equation (6.13). The spin sums which finally give the *Elko* propagator are presented in equations (5.55) and (5.56), with the resulting covariant amplitude given in equation (6.20). The propagator, in the absence of a preferred direction, is written in equation (6.28). It is to be contrasted with the Dirac propagator recorded in equation (6.30). The non-locality anticommutators are found in equations (8.8), (8.20), and in a remark which encloses equation (8.26). An estimate of the sensitivity of the *Elko* non-locality to its mass can be obtained from (8.25). The non-locality as manifest in $\mathcal{T}\left(\eta(x)\stackrel{\neg}{\eta}(y)\eta(z)\stackrel{\neg}{\eta}(w)\right)$ is discusssed around equation (8.45). The quartic self-interaction of *Elko* is to be read at equation (9.2), while equation (9.3) provides Higgs–*Elko* interaction. A remarkably simple equation which connects the number of stars in a typical galaxy with three relevant elementary particle masses, including that of *Elko*, is seen at equation (10.20). It defines the physical viability for the identification of the *Elko* framework with dark matter. Equations (10.21) and (10.25) give analytical expressions for possible *Elko* masses. The fractional *Elko* contribution to the total cosmic energy density, Ω_ς, is given by equation (10.15). Although interactions with gravity were not our main concern, a brief study of Elko particles in a Thirring–Lense gravitational background revealed the interesting possibility of oscillations between self- and anti-self-conjugate states, with an oscillation frequency given by equation (11.5). A hint for non-commutative momentum space, and its possible dependence on representation space (i.e., spin content of the probing test particle), is suggested by the results contained in equations (12.7) and (12.8).

12.6 Summary

The unexpected theoretical result of this paper is: a fermionic quantum field based on dual-helicity eigenspinors[50] of the spin-1/2 charge conjugation operator carries mass dimension one, and not three-halves. This circumstance forbids a large class of interactions with gauge and matter fields of the Standard Model, while allowing for an interaction with the Higgs field. In addition, owing to mass dimension one, the introduced field is endowed with a quartic self-interaction. This suggests a first-principle identification of the new field with dark matter. Thus, regarding the question we asked in the introduction as to what constitutes dark matter, we have provided a new possible answer, namely *Elko*. The question on dark energy shall perhaps be the subject of a subsequent paper. The indicated interaction calls for some very specific properties for a gravitationally induced collapse of a galactic-mass cloud of the new particles. In particular it asks for a supernova-like

[50] We have called these *Elko* after their German name.

explosion for the collapsing cloud. A semi-quantitative argument yields three different values for the mass of the new dark matter candidate. These values are 3 keV, 1–1.2 MeV, and 0.5 TeV as lower bounds. The first of these values arises if the quartic self-interaction is held responsible for the explosion, while the second value results from Higgs being behind the phenomena, and the last value results from the possibility of Planck-scale physics. Considerations on the relic abundance of *Elko*, and various astrophysical observations, suggest a 20 MeV mass for *Elko* particles.

We conclude this long exposition with two remarks.

(a) The non-locality that appears in *Elko* theory carries no free parameters and is governed entirely by the *Elko* mass and resides in the well established spacetime symmetries. Therefore, for a theoretical physicist it may serve as a fertile playing ground for examining phenomenological consequences of non-locality.

(b) Ordinarily, dark matter is postulated not to carry any Standard Model interactions. The presented theory does not postulate this absence as an input but requires it in the sense made precise above.

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A. Appendix: Auxiliary details

A.1 The $\phi^{\pm}_{\mathrm{L}}(\mathbf{0})$

Representing the unit vector along $\mathbf{p}$, as

$$\widehat{\mathbf{p}} = \Big(\sin(\theta)\cos(\phi),\, \sin(\theta)\sin(\phi),\, \cos(\theta)\Big)\,, \tag{A.1}$$

the $\phi_{\rm L}^{\pm}(\mathbf{0})$ take the explicit form

$$\phi_{\rm L}^{+}(\mathbf{0}) = \sqrt{m}\mathrm{e}^{\mathrm{i}\vartheta_1} \begin{pmatrix} \cos(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2} \\ \sin(\theta/2)\mathrm{e}^{\mathrm{i}\phi/2} \end{pmatrix}, \tag{A.2}$$

$$\phi_{\rm L}^{-}(\mathbf{0}) = \sqrt{m}\mathrm{e}^{\mathrm{i}\vartheta_2} \begin{pmatrix} \sin(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2} \\ -\cos(\theta/2)\mathrm{e}^{\mathrm{i}\phi/2} \end{pmatrix}. \tag{A.3}$$

In this paper we take ϑ_1 and ϑ_2 to be zero.

For the evaluation of spin sums (cf appendix B.4) the following identities are useful:

$$\sigma_2\phi_{\rm L}^{+}(\mathbf{0}) = \sqrt{m} \begin{pmatrix} -\mathrm{i}\sin(\theta/2)\mathrm{e}^{\mathrm{i}\phi/2} \\ \mathrm{i}\cos(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2} \end{pmatrix}, \tag{A.4}$$

$$\sigma_2\phi_{\rm L}^{-}(\mathbf{0}) = \sqrt{m} \begin{pmatrix} \mathrm{i}\cos(\theta/2)\mathrm{e}^{\mathrm{i}\phi/2} \\ \mathrm{i}\sin(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2} \end{pmatrix}, \tag{A.5}$$

as they imply

$$(\phi_{\rm L}^{\pm}(\mathbf{0}))^{\dagger}\sigma_2(\phi_{\rm L}^{\pm}(\mathbf{0}))^{*} = 0\,, \tag{A.6}$$

$$(\phi_{\rm L}^{\pm}(\mathbf{0}))^{\dagger}\sigma_2(\phi_{\rm L}^{\mp}(\mathbf{0}))^{*} = \pm\mathrm{i}\,. \tag{A.7}$$

Additional helpful relations are

$$(\phi_{\rm L}^{\pm}(\mathbf{0}))^{\dagger}(\phi_{\rm L}^{\pm}(\mathbf{0})) = 1\,, \tag{A.8}$$

$$(\phi_{\rm L}^{\pm}(\mathbf{0}))^{\dagger}(\phi_{\rm L}^{\mp}(\mathbf{0})) = 0\,. \tag{A.9}$$

A.2 Helicity properties of $\Theta(\phi_{\rm L}^{\pm}(\mathbf{0}))^{*}$

Complex conjugating equation (3.7) gives

$$\boldsymbol{\sigma}^{*}\cdot\widehat{\mathbf{p}}\ \left[\phi_{\rm L}^{\pm}(\mathbf{0})\right]^{*} = \pm\ \left[\phi_{\rm L}^{\pm}(\mathbf{0})\right]^{*}. \tag{A.10}$$

Substituting for $\boldsymbol{\sigma}^{*}$ from equation (3.2) then results in

$$\Theta\boldsymbol{\sigma}\Theta^{-1}\cdot\widehat{\mathbf{p}}\ \left[\phi_{\rm L}^{\pm}(\mathbf{0})\right]^{*} = \mp\ \left[\phi_{\rm L}^{\pm}(\mathbf{0})\right]^{*}. \tag{A.11}$$

But $\Theta^{-1} = -\Theta$. So

$$-\,\Theta\boldsymbol{\sigma}\Theta\cdot\widehat{\mathbf{p}}\ \left[\phi_{\rm L}^{\pm}(\mathbf{0})\right]^{*} = \mp\ \left[\phi_{\rm L}^{\pm}(\mathbf{0})\right]^{*}. \tag{A.12}$$

Or, equivalently,

$$\Theta^{-1}\boldsymbol{\sigma}\Theta\cdot\widehat{\mathbf{p}}\ \left[\phi_{\rm L}^{\pm}(\mathbf{0})\right]^{*} = \mp\ \left[\phi_{\rm L}^{\pm}(\mathbf{0})\right]^{*}. \tag{A.13}$$

Finally, left-multiplying both sides of the preceding equation by Θ, and moving Θ through $\widehat{\mathbf{p}}$, yields equation(3.8).

B. Appendix: Elkology details

B.1 Bi-orthonormality relations for $\boldsymbol{\lambda}(\mathbf{p})$ spinors

On setting ϑ_1 and ϑ_2 to be zero — a fact that we explicitly note [41, 139] — we find the following *bi-orthonormality* relations for the self-conjugate spinors:

$$\overline{\lambda}^{\mathrm{S}}_{\{-,+\}}(\mathbf{p})\lambda^{\mathrm{S}}_{\{-,+\}}(\mathbf{p}) = 0\,, \qquad \overline{\lambda}^{\mathrm{S}}_{\{-,+\}}(\mathbf{p})\lambda^{\mathrm{S}}_{\{+,-\}}(\mathbf{p}) = +2\mathrm{i}m\,, \tag{B.1}$$

$$\overline{\lambda}^{\mathrm{S}}_{\{+,-\}}(\mathbf{p})\lambda^{\mathrm{S}}_{\{-,+\}}(\mathbf{p}) = -2\mathrm{i}m\,, \qquad \overline{\lambda}^{\mathrm{S}}_{\{+,-\}}(\mathbf{p})\lambda^{\mathrm{S}}_{\{+,-\}}(\mathbf{p}) = 0\,. \tag{B.2}$$

Their counterpart for anti-self-conjugate spinors reads

$$\overline{\lambda}^{\mathrm{A}}_{\{-,+\}}(\mathbf{p})\lambda^{\mathrm{A}}_{\{-,+\}}(\mathbf{p}) = 0\,, \qquad \overline{\lambda}^{\mathrm{A}}_{\{-,+\}}(\mathbf{p})\lambda^{\mathrm{A}}_{\{+,-\}}(\mathbf{p}) = -2\mathrm{i}m\,, \tag{B.3}$$

$$\overline{\lambda}^{\mathrm{A}}_{\{+,-\}}(\mathbf{p})\lambda^{\mathrm{A}}_{\{-,+\}}(\mathbf{p}) = +2\mathrm{i}m\,, \qquad \overline{\lambda}^{\mathrm{A}}_{\{+,-\}}(\mathbf{p})\lambda^{\mathrm{A}}_{\{+,-\}}(\mathbf{p}) = 0\,, \tag{B.4}$$

while all combinations of the type $\overline{\lambda}^{\mathrm{A}}(\mathbf{p})\lambda^{\mathrm{S}}(\mathbf{p})$ and $\overline{\lambda}^{\mathrm{S}}(\mathbf{p})\lambda^{\mathrm{A}}(\mathbf{p})$ identically vanish. We take note that the bi–orthogonal norms of the *Elko* are intrinsically *imaginary.* The associated completeness relation is

$$-\frac{1}{2\mathrm{i}m}\Bigg(\left[\lambda^{\mathrm{S}}_{\{-,+\}}(\mathbf{p})\overline{\lambda}^{\mathrm{S}}_{\{+,-\}}(\mathbf{p}) - \lambda^{\mathrm{S}}_{\{+,-\}}(\mathbf{p})\overline{\lambda}^{\mathrm{S}}_{\{-,+\}}(p)\right] \\ - \left[\lambda^{\mathrm{A}}_{\{-,+\}}(\mathbf{p})\overline{\lambda}^{\mathrm{A}}_{\{+,-\}}(\mathbf{p}) - \lambda^{\mathrm{A}}_{\{+,-\}}(\mathbf{p})\overline{\lambda}^{\mathrm{A}}_{\{-,+\}}(\mathbf{p})\right]\Bigg) = \mathbb{I}\,. \tag{B.5}$$

B.2 The $\boldsymbol{\rho}(\mathbf{p})$ spinors

Now, $(1/2, 0)\oplus(0, 1/2)$ is a four-dimensional representation space. Therefore, there cannot be more than four independent spinors. Consistent with this observation, we find that the $\rho(\mathbf{p})$ spinors are related to the $\lambda(\mathbf{p})$ spinors through the following identities:

$$\rho^{\mathrm{S}}_{\{+,-\}}(\mathbf{p}) = +\,\mathrm{i}\lambda^{\mathrm{A}}_{\{+,-\}}(\mathbf{p})\,, \quad \rho^{\mathrm{S}}_{\{-,+\}}(\mathbf{p}) = -\,\mathrm{i}\lambda^{\mathrm{A}}_{\{-,+\}}(\mathbf{p}), \tag{B.6}$$

$$\rho^{\mathrm{A}}_{\{+,-\}}(\mathbf{p}) = -\,\mathrm{i}\lambda^{\mathrm{S}}_{\{+,-\}}(\mathbf{p})\,, \quad \rho^{\mathrm{A}}_{\{-,+\}}(\mathbf{p}) = +\,\mathrm{i}\lambda^{\mathrm{S}}_{\{-,+\}}(\mathbf{p})\,. \tag{B.7}$$

Using these identities, one may immediately obtain the bi-orthonormality and completeness relations for the $\rho(\mathbf{p})$ spinors. In the massless limit, $\rho^{\mathrm{S}}_{\{+,-\}}(\mathbf{p})$ and $\rho^{\mathrm{A}}_{\{+,-\}}(\mathbf{p})$ *identically vanish.* A particularly simple orthonormality, as opposed to bi-orthonormality, relation exists between the $\lambda(\mathbf{p})$ and $\rho(\mathbf{p})$ spinors:

$$\overline{\lambda}^{\mathrm{S}}_{\{-,+\}}(\mathbf{p})\rho^{\mathrm{A}}_{\{-,+\}}(\mathbf{p}) = -2m = \overline{\lambda}^{\mathrm{A}}_{\{-,+\}}(\mathbf{p})\rho^{\mathrm{S}}_{\{-,+\}}(\mathbf{p}) \tag{B.8}$$

$$\overline{\lambda}^{\mathrm{S}}_{\{+,-\}}(\mathbf{p})\rho^{\mathrm{A}}_{\{+,-\}}(\mathbf{p}) = -2m = \overline{\lambda}^{\mathrm{A}}_{\{+,-\}}(\mathbf{p})\rho^{\mathrm{S}}_{\{+,-\}}(\mathbf{p}). \tag{B.9}$$

An associated completeness relation also exists, and it reads

$$-\frac{1}{2m}\Bigg(\left[\lambda^{\mathrm{S}}_{\{-,+\}}(\mathbf{p})\overline{\rho}^{\mathrm{A}}_{\{-,+\}}(\mathbf{p}) + \lambda^{\mathrm{S}}_{\{+,-\}}(\mathbf{p})\overline{\rho}^{\mathrm{A}}_{\{+,-\}}(p)\right] \\ + \left[\lambda^{\mathrm{A}}_{\{-,+\}}(\mathbf{p})\overline{\rho}^{\mathrm{S}}_{\{-,+\}}(\mathbf{p}) + \lambda^{\mathrm{A}}_{\{+,-\}}(\mathbf{p})\overline{\rho}^{\mathrm{S}}_{\{+,-\}}(\mathbf{p})\right]\Bigg) = \mathbb{I}\,. \tag{B.10}$$

The results of this section are in the spirit of [38, 39, 41, 139].

The completeness relation (B.5) confirms that a physically complete theory of fundamentally neutral particle spinors must incorporate the self as well as anti-self-conjugate spinors. However, one has a choice. One may either work with the set $\{\lambda^{\rm S}(\mathbf{p}), \lambda^{\rm A}(\mathbf{p}\})$, or with the physically and mathematically equivalent set, $\{\rho^{\rm S}(\mathbf{p}), \rho^{\rm A}(\mathbf{p})\}$. One is also free to choose some appropriate combinations of neutral particle spinors from these two sets.

B.3 Elko in the Majorana realization

The $\lambda^{\rm S,A}(\mathbf{p})$ obtained above are in Weyl realization (subscripted by W). In Majorana realization (subscripted by M) these spinors are given by:

$$\lambda^{\rm S,A}_{\rm M}(\mathbf{p}) = \mathcal{S}\,\lambda^{\rm S,A}_{\rm W}(\mathbf{p})\,, \tag{B.11}$$

where

$$\mathcal{S} = \frac{1}{2}\left(\begin{array}{cc} \mathbb{I}+{\rm i}\Theta & \mathbb{I}-{\rm i}\Theta \\ -\left(\mathbb{I}-{\rm i}\Theta\right) & \mathbb{I}+{\rm i}\Theta \end{array}\right)\,. \tag{B.12}$$

Calculations show that the $\lambda^{\rm S}_{\rm M}(\mathbf{p})$ are real, while the $\lambda^{\rm A}_{\rm M}(\mathbf{p})$ are imaginary.

B.4 Spin sums

The evaluation of the spin sums is straightforward. We perform it here explicitly for the self-dual spinors and sketch briefly the result for the anti-self-dual ones. The definition (3.17) together with (B.7), (3.12) and (3.13) yields

$$\sum_{\alpha=\{-,+\},\{+,-\}} \lambda^{\rm S}_\alpha(\mathbf{p})\stackrel{\neg{\rm S}}{\lambda_\alpha}(\mathbf{p}) = {\rm i}\,\frac{(E+m)^2-p^2}{2m(E+m)}\,\mathcal{S}\,, \tag{B.13}$$

with

$$\mathcal{S} := \left(\lambda^{\rm S}_{\{-,+\}}(\mathbf{0})\bar\lambda^{\rm S}_{\{+,-\}}(\mathbf{0}) - \lambda^{\rm S}_{\{+,-\}}(\mathbf{0})\bar\lambda^{\rm S}_{\{-,+\}}(\mathbf{0})\right)\,. \tag{B.14}$$

Note that (B.14) contains only 'ordinary' Dirac bars. By virtue of the identities

$$\phi^+_{\rm L}(\mathbf{0})(\phi^-_{\rm L}(\mathbf{0}))^\dagger - \phi^-_{\rm L}(\mathbf{0})(\phi^+_{\rm L}(\mathbf{0}))^\dagger = -{\rm i}m\mathcal{A}^{\rm S} \tag{B.15}$$

and

$$\sigma_2(\mathcal{A}^{\rm S})^*\sigma_2 = -\mathcal{A}^{\rm S} \tag{B.16}$$

one obtains

$$\mathcal{S} = -{\rm i}m\left(\begin{array}{cc} \mathbb{I} & \mathcal{A}^{\rm S} \\ \mathcal{A}^{\rm S} & \mathbb{I}\end{array}\right)\,. \tag{B.17}$$

In conjunction with the dispersion relation (12.5), the result (5.18) is finally produced.

The definition (3.18) together with (B.6), (3.14) and (3.15) establishes the same result as in (B.13) and (B.14) with superscript S replaced by A. Consequently, the whole expression acquires an overall sign and $\mathcal{A}^{\rm S}$ has to be replaced by $\mathcal{A}^{\rm A}$. Having inserted $\mathcal{A}^{\rm A} = -\mathcal{A}^{\rm S}$ the result is displayed in (5.19).

In a similar fashion the following spin sums may be evaluated:

$$\sum_{\alpha=\{-,+\},\{+,-\}} \lambda^{\mathrm{S}}_\alpha(\mathbf{p})\left(\lambda^{\mathrm{S}}_\alpha(\mathbf{p})\right)^\dagger = (E-p)\left(\mathbb{I}+\mathcal{G}\right)\,, \tag{B.18}$$

and

$$\sum_{\alpha=\{-,+\},\{+,-\}} \lambda^{\mathrm{A}}_\alpha(\mathbf{p})\left(\lambda^{\mathrm{A}}_\alpha(\mathbf{p})\right)^\dagger = (E+p)\left(\mathbb{I}-\mathcal{G}\right)\,, \tag{B.19}$$

with $\mathcal{G}$ as defined in (5.54).

The 'twisted' spin sums relevant to non-locality turns out as

$$\sum_\beta \left[\lambda^{\mathrm{S}}_\beta(\mathbf{p})\left(\lambda^{\mathrm{A}}_\beta(\mathbf{p})\right)^{\mathrm{T}} + \lambda^{\mathrm{S}}_\beta(-\mathbf{p})\left(\lambda^{\mathrm{A}}_\beta(-\mathbf{p})\right)^{\mathrm{T}}\right]$$

$$= 2\begin{pmatrix} e^{-\mathrm{i}\phi}\,p\cos(\theta) & p\sin(\theta) & 0 & -\mathrm{i}\,E \\ p\sin(\theta) & -\mathrm{e}^{+\mathrm{i}\phi}\,p\cos(\theta) & \mathrm{i}\,E & 0 \\ 0 & -\mathrm{i}\,E & -\mathrm{e}^{-\mathrm{i}\phi}p\cos(\theta) & -p\sin(\theta) \\ \mathrm{i}\,E & 0 & -p\sin(\theta) & \mathrm{e}^{+\mathrm{i}\phi}\,p\cos(\theta) \end{pmatrix}, \tag{B.20}$$

and

$$\sum_\beta \left[\left(\overset{\neg\mathrm{S}}{\lambda_\beta}(\mathbf{p})\right)^\dagger \overset{\neg\mathrm{A}}{\lambda_\beta}(\mathbf{p}) + \left(\overset{\neg\mathrm{S}}{\lambda_\beta}(-\mathbf{p})\right)^\dagger \overset{\neg\mathrm{A}}{\lambda_\beta}(-\mathbf{p})\right]$$

$$= 2\begin{pmatrix} \sqrt{p^2+m^2} & 0 & \mathrm{i}\,p\sin(\theta) & -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\phi}\,p\cos(\theta) \\ 0 & \sqrt{p^2+m^2} & -\mathrm{i}\,\mathrm{e}^{+\mathrm{i}\phi}\,p\cos(\theta) & -\mathrm{i}\,p\sin(\theta) \\ \mathrm{i}\,p\sin(\theta) & -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\phi}\,p\cos(\theta) & -\sqrt{p^2+m^2} & 0 \\ -\mathrm{i}\,\mathrm{e}^{+\mathrm{i}\phi}\,p\cos(\theta) & -\mathrm{i}\,p\sin(\theta) & 0 & -\sqrt{p^2+m^2} \end{pmatrix}. \tag{B.21}$$

Finally, the identities

$$\left(\lambda^{\mathrm{S/A}}_{\{-,+\}}(\mathbf{p})\right)^\dagger \lambda^{\mathrm{S/A}}_{\{+,-\}}(\mathbf{p}) = 0\,, \tag{B.22}$$

$$\left(\lambda^{\mathrm{S/A}}_{\{+,-\}}(\mathbf{p})\right)^\dagger \lambda^{\mathrm{S/A}}_{\{-,+\}}(\mathbf{p}) = 0\,, \tag{B.23}$$

and

$$\left(\lambda^{\mathrm{S/A}}_{\{-,+\}}(\mathbf{p})\right)^\dagger \lambda^{\mathrm{S/A}}_{\{-,+\}}(\mathbf{p}) = 2(E-p)\,, \tag{B.24}$$

$$\left(\lambda^{\mathrm{S/A}}_{\{+,-\}}(\mathbf{p})\right)^\dagger \lambda^{\mathrm{S/A}}_{\{+,-\}}(\mathbf{p}) = 2(E+p)\,, \tag{B.25}$$

may be useful in various contexts.

B.5 Distributional part of $\{\boldsymbol{\eta}, \boldsymbol{\eta}\}$

An integral such as that in (8.16) can be evaluated by methods described in [140] in the context of the Fourier transformation of $\theta(x)x^\lambda$. The general result

$$\lim_{\epsilon\to 0}\int_0^\infty x^\lambda \mathrm{e}^{\mathrm{i}(k+\mathrm{i}\epsilon)x}\,\mathrm{d}x = \mathrm{i}\mathrm{e}^{\mathrm{i}\lambda\pi/2}\Gamma(\lambda+1)\lim_{\epsilon\to 0}\left(k+\mathrm{i}\epsilon\right)^{-\lambda-1} \tag{B.26}$$

with $\lambda = 1$ can be applied to obtain

$$\int_0^\infty x \sin kx \, \mathrm{d}x := \lim_{\epsilon\to 0} \frac{1}{2\mathrm{i}} \int_0^\infty x \left(\mathrm{e}^{\mathrm{i}(k+\mathrm{i}\epsilon)x} - \mathrm{e}^{-\mathrm{i}(k-\mathrm{i}\epsilon)x} \right) = \lim_{\epsilon\to 0} \frac{1}{2\mathrm{i}} \left(\frac{1}{k^2 - \mathrm{i}\epsilon} - \frac{1}{k^2 + \mathrm{i}\epsilon} \right)$$
$$= \lim_{\epsilon\to 0} \frac{1}{2\mathrm{i}} \left(P\left(\frac{1}{k^2}\right) + \mathrm{i}\pi\delta(k^2) - P\left(\frac{1}{k^2}\right) + \mathrm{i}\pi\delta(k^2) \right) = \pi\delta(k^2) \,. \tag{B.27}$$

The symbol P denotes the principal value. In our case the quantity k is nothing but the radius r. An alternative representation of (B.27) follows from

$$\int_0^\infty x \sin kx \, \mathrm{d}x := -\frac{\mathrm{d}}{\mathrm{d}k} \lim_{\epsilon\to 0} \frac{1}{2} \int_0^\infty \left(\mathrm{e}^{\mathrm{i}(k+\mathrm{i}\epsilon)x} - \mathrm{e}^{-\mathrm{i}(k-\mathrm{i}\epsilon)x} \right) = \frac{\mathrm{d}}{\mathrm{d}k} \lim_{\epsilon\to 0} \frac{\mathrm{i}}{2} \left(\frac{1}{k - \mathrm{i}\epsilon} - \frac{1}{k + \mathrm{i}\epsilon} \right)$$
$$= \frac{\mathrm{d}}{\mathrm{d}k} \lim_{\epsilon\to 0} \frac{\mathrm{i}}{2} \left(P\left(\frac{1}{k}\right) + \mathrm{i}\pi\delta(k) - P\left(\frac{1}{k}\right) + \mathrm{i}\pi\delta(k) \right) = -\pi\delta'(k) \,. \tag{B.28}$$

B.6 On the ϕ dependence of $\mathcal{O}$ for Elko and non-standard dispersion relations

The matrix $\mathcal{G}$ depends on a direction $\mathbf{n}$ which is orthogonal to the direction of propagation, $\widehat{\mathbf{p}}$, but it is independent from $|\mathbf{p}|$ and p_0, in contrast to the Dirac case. A different way of writing the projection operators is

$$\frac{1}{2}(\mathbb{I} \pm \mathcal{G}) = \frac{\mathbb{I} \pm \gamma^5 \gamma_\mu n^\mu}{2} \,, \tag{B.29}$$

with $n_\mu = (0, \mathbf{n})$. We emphasize that $\mathbf{n} = (1/\sin\theta)\mathrm{d}\widehat{\mathbf{p}}/\mathrm{d}\phi$ is not independent from $\mathbf{p}$ as it depends on ϕ.

In the light of this angular dependence we consider the possibility of a boost operator different from the standard one, implying in general also a non-standard dispersion relation. To this end let us replace (2.2) and (2.3) by

$$\kappa^{(1/2,0)} = \exp A \,, \quad \kappa^{(0,1/2)} = \exp B \,, \tag{B.30}$$

with some as yet unspecified operators A, B. The exponential representation has been chosen in order to make invertibility manifest, but for specific cases other representations might be more useful. All identities derived in section 5.3 still hold. As the operator $\mathcal{O}$ has block form with mutually commuting non-singular entries, its determinant is given by

$$\det \mathcal{O} = \det\left(\mathbb{I} - \mathcal{D}\mathcal{D}^{-1}\right) = 0 \,, \tag{B.31}$$

with $\mathcal{D}$ as defined in (5.30). Thus, the determinant of $\mathcal{O}$ vanishes without implying further restrictions.

Regarding the multiplicity of the dispersion relations it should be noted that the matrix $\mathcal{O}$ in (5.33) maximally has half rank because the lower block linearly depends from the upper one. If $\mathcal{D} \propto \mathbb{I}$ the rank of $\mathcal{O}$ is 1, else it is 2. The first case is trivial and may arise only for very special choices of A, B and $\mathcal{A}$. Thus, generically there will be either one dispersion relation with multiplicity 2 as in (12.5) or two dispersion relations with multiplicity 1 as in (12.6). Clearly, the explicit form of the dispersion relations will depend on the choice of the boost operator, but *not* on the matrix $\mathcal{A}$. For the standard choice (2.2), (2.3) only the standard dispersion relation (12.5) appears.

Finally, the question will be addressed to what extent *Elko* particles may probe non-commutativity of energy–momentum space or deformations of the Lorentz group different from the way Dirac particles do. Such a difference, if any, can be traced back to the behavior of the matrix $\mathcal{A}$ encoding the CPT properties which is proportional to the unit matrix only for Dirac particles. The dispersion relation is independent from it, but the spin sum operator $\mathcal{O}$ is sensitive to it. Consequently, *Elko* particles may probe aspects of non-standard dispersion relations in a way different from Dirac particles.

For the sake of concreteness we suppose $A = A_\mu \sigma^\mu$ and $B = B_\mu \sigma^\mu$ with $\sigma^\mu = (\mathbb{I}, \boldsymbol{\sigma})$. It is useful to introduce an adapted orthogonal Dreibein $\widehat{\mathbf{p}}, \mathbf{n} = (1/\sin\theta) d\widehat{\mathbf{p}}/d\phi, \mathbf{l} := \mathbf{n} \times \widehat{\mathbf{p}} = \mathrm{d}\widehat{\mathbf{p}}/\mathrm{d}\theta$ and to decompose $\boldsymbol{\sigma}$ with respect to it. Note that in the Dirac case $\mathcal{A}$ trivially commutes with all these projections, while for *Elko* we obtain

$$[\mathcal{A}, \sigma^0] = 0\,, \qquad \{\mathcal{A}, \boldsymbol{\sigma}\cdot\mathbf{p}\} = 0\,, \qquad [\mathcal{A}, \boldsymbol{\sigma}\cdot\mathbf{n}] = 0\,, \qquad \{\mathcal{A}, \mathbf{l}\cdot\mathbf{p}\} = 0\,. \tag{B.32}$$

Therefore, for Dirac particles $\mathcal{D} = \exp(A_\mu\sigma^\mu)\exp(B_\mu\sigma^\mu)\mathcal{A}$, while for *Elko* particles $\mathcal{D} = \exp(A_\mu\sigma^\mu)\exp(\tilde{B}_\mu\sigma^\mu)\mathcal{A}$, where $\tilde{B}$ can be derived from B by virtue of (B.32). Because B typically has a non-vanishing $\widehat{\mathbf{p}}$ component, $\tilde{B} \neq B$ in general. A possibility for non-standard boost operators which can be considered as 'natural' in the context of *Elko* particles has been addressed in section 12.

B.7 Some other anticommutators in the context of non-locality discussion

Here we collect the anticommutators $\{\eta^\dagger(\mathbf{x},t), \eta^\dagger(\mathbf{x}',t)\}$ and $\{\stackrel{\neg}{\eta}(\mathbf{x},t), \stackrel{\neg}{\eta}(\mathbf{x}',t)\}$. The former just follows from (8.20) by Hermitean conjugation and thus provides

$$\Big\langle\ \Big|\Big\{\eta^\dagger(\mathbf{x},t), \eta^\dagger(\mathbf{x}',t)\Big\}\Big|\ \Big\rangle = -\frac{1+mr}{4\,m\,\pi r^3} e^{-mr}\gamma^0\gamma^1 - \frac{1}{2\,m\,\pi r^2}\delta(r^2)\gamma^2\gamma^5\,, \tag{B.33}$$

because all γ matrices are Hermitean and anticommute with each other. It should be noted that (B.33) is just the negative of (8.20). By virtue of the identities

$$\Big\{\stackrel{\neg}{\eta}(\mathbf{x},t), \stackrel{\neg}{\eta}(\mathbf{x}',t)\Big\} \sim \gamma^0 \Big\{\eta^\dagger(\mathbf{x},t), \eta^\dagger(\mathbf{x}',t)\Big\}\gamma^0 \sim -\gamma^0\left\{\eta(\mathbf{x},t), \eta(\mathbf{x}',t)\right\}\gamma^0\,, \tag{B.34}$$

where $\sim$ means equivalence of corresponding vacuum expectation values, one obtains after trivial rearrangements of γ matrices,

$$\Big\langle\ \Big|\Big\{\stackrel{\neg}{\eta}(\mathbf{x},t), \stackrel{\neg}{\eta}(\mathbf{x}',t)\Big\}\Big|\ \Big\rangle = \frac{1+mr}{4\,m\,\pi r^3} e^{-mr}\gamma^0\gamma^1 - \frac{1}{2\,m\,\pi r^2}\delta(r^2)\gamma^2\gamma^5\,. \tag{B.35}$$

This result exhibits the same behavior of the distributional part as (B.33) and the same behavior of the remaining part as (8.20).

Note added in proof. While this paper was being proof read, [141] appeared, which places *Elko* as Lounesto's class 5 spinors. It further emphasizes its differences and similarities with the Majorana spinors. The authors also confirm our result contained in equation (4.16).

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